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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} *anniversary*

The generalization of Voronovskaja's theorem for exponential operators

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ABSTRACT. In this paper we will demonstrate a Voronovskaja's type general theorem for exponential operators. By particularization, we obtain Voronovskaja's type theorems for the different operators.

1. INTRODUCTION

We set $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let a and b such that $-\infty \le a < b \le \infty$. In this paper we consider the notation

$$I(a,b) = \begin{cases} [a,b], & \text{if} \quad a, b \in \mathbb{R} \\ (-\infty, b], & \text{if} \quad a = -\infty, b \in \mathbb{R} \\ [a, \infty)\,, & \text{if} \quad a \in \mathbb{R}, b = \infty \\ (-\infty, \infty) = \mathbb{R}, & \text{if} \quad a = -\infty, b = \infty\,. \end{cases} \tag{1.1}$$

Let $n \in \mathbb{N}$. The kernel $W_n : I(a,b) \times I(a,b) \to \mathbb{R}$ satisfies

$$W_n(x,t) \ge 0 \tag{1.2}$$

for any $(x,t) \in I(a,b) \times I(a,b)$,

$$\int_a^b W_n(x,t)dt = 1 \tag{1.3}$$

for any $x \in I(a,b)$ and

$$\frac{\partial}{\partial x} W_n(x,t) = \frac{n(t-x)}{p(x)} W_n(x,t) \tag{1.4}$$

for any $(x,t) \in I(a,b) \times I(a,b)$, where $p(x)$ is polynomial in x and $p(x)$ is strictly positive for any $x \in I(a,b)$.

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We define the operators $S_n : \mathcal{F}(p) \to C(\mathbb{R})$, for any function $f \in \mathcal{F}(p)$ by

$$(S_n f)(x) = \int\limits_a^b W_n(x,t) f(t) dt \tag{1.5}$$

for any $x \in I(a,b)$, where $\mathcal{F}(p) = \big\{ f : I(a,b) \to \mathbb{R} \,|\, \int\limits_a^b W_n(x,t)f(t)dt < \infty$ for any $x \in I(a,b)$ and for any $n \in \mathbb{N} \big\}$.

The operators $(S_n)_{n \geq 1}$ are introduced and are studied by C. P. May in the paper [8]. These operators are referred to us like exponential operators.

Let $n, m \in \mathbb{N}_0$, $n \neq 0$. The m-th centered order moment is defined by

$$A_{n,m}(x) = n^m \int\limits_a^b W_n(x,t)(t-x)^m dt \tag{1.6}$$

for any $x \in I(a,b)$.

The results contained in the following lemmas are well known (see [1] or [17]).

Lemma 1.1. *We have that*

$$A_{n,0}(x) = 1, \tag{1.7}$$

$$A_{n,1}(x) = 0 \tag{1.8}$$

and

$$A_{n,2}(x) = np\,(x) \tag{1.9}$$

for any $x \in I(a,b)$ and for any $n \in \mathbb{N}$.

Lemma 1.2. *Let $n, m \in \mathbb{N}_0$, $n \neq 0$ and $x \in I(a,b)$. Then the m-th centered order moment $A_{n,m}(x)$ is a $\left[\dfrac{m}{2}\right]$ degree polynomial in n.*

Consequence 1.1. For any $m \in \mathbb{N}_0$ and for any $x \in I(a,b)$, it results that exists

$$\lim_{n\to\infty} \frac{A_{n,m}(x)}{n^{[\frac{m}{2}]}}$$

and

$$\lim_{n\to\infty} \frac{A_{n,m}(x)}{n^{[\frac{m}{2}]}} = l_m(x) \in \mathbb{R}. \tag{1.10}$$

Proof. It results from Lemma 1.2. □

Lemma 1.3. *For any $m, n \in \mathbb{N}_0$, $n \neq 0$, the relation*

$$A_{n,m+1}(x) = mnp\,(x) A_{n,m-1}(x) + p\,(x) \frac{d}{dx} A_{n,m}(x) \tag{1.11}$$

holds for any $x \in I(a,b)$.

Consequence 1.2. We have that

$$A_{n,3}(x) = np'(x)p\,(x) \tag{1.12}$$

and

$$A_{n,4}(x) = 3n^2p^2(x) + np''(x)p^2(x) + n[p'(x)]^2 p\,(x), \tag{1.13}$$

for any $x \in I(a,b)$ and for any $n \in \mathbb{N}$.

Proof. It results from Lemma 1.3 and Lemma 1.1. □

Lemma 1.4. *Let $n, m \in \mathbb{N}_0$, $n \neq 0$ and $x \in I(a,b)$. Then the m-th centered order moment $A_{n,m}(x)$ is a polynomial in n.*

Consequence 1.3. Let $n \in \mathbb{N}$ and $x \in I(a,b)$. Then

$$A_{n,m}(x) = n^{[\frac{m}{2}]} p_m(x) + q_{n,m}(x) \tag{1.14}$$

for any $m \in \mathbb{N}_0$, where $p_m(x)$ is a polynomial in x and the degree of $q_{n,m}(x)$ polynomial in n is strictly smaller than $\left[\dfrac{m}{2}\right]$.

Proof. By induction from m, taking into account by Lemma 1.2, Lemma 1.3 and Lemma 1.4. □

Consequence 1.4. For any $m \in \mathbb{N}_0$ and any compact set K, $K \subset I(a,b)$, there exits $k_m(K) \in \mathbb{R}$ depending on K and m, there exists $n_m \in \mathbb{N}$ depending on m such that

$$\frac{A_{n,m}(x)}{n^{[\frac{m}{2}]}} \leq k_m(K) \tag{1.15}$$

for any $n \in \mathbb{N}$, $n \geq n_m$ and for any $x \in K$.

Proof. From Consequence 1.1 and Consequence 1.3 it results that $l_m(x)$ is a polynomial in x. Then there exists $n_m \in \mathbb{N}$ such that $\dfrac{A_{n,m}(x)}{n^{[\frac{m}{2}]}} < l_m(x)+1$ for any $n \in \mathbb{N}$, $n \geq n_m$ and because K is a compact set and nothing $\sup\limits_{x \in K} [l_m(x)+1] = k_m(K)$, then (1.15) holds. □

2. Preliminaries

In this section, we recall some notions and results which we will use in this article.

Let $n \in \mathbb{N}$ and $B_n : C([0,1]) \to C([0,1])$ the Bernstein operators, defined for any function $f \in C([0,1])$ by

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{2.1}$$

where $p_{n,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \tag{2.2}$$

for any $x \in [0,1]$ and any $k \in \{0, 1, \ldots, n\}$.

Let $n \in \mathbb{N}$ and the operators $S_n : C_2([0,\infty)) \to C([0,\infty))$ defined for any function $f \in C_2([0,\infty))$ by

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{2.3}$$

for any $x \in [0,\infty)$, where $C_2([0,\infty)) = \left\{ f \in C([0,\infty)) : \lim\limits_{x\to\infty} \dfrac{f(x)}{1+x^2} \text{ exists and is finite} \right\}$. The operators $(S_n)_{n\geq 1}$ are named Mirakjan-Favard-Szász operators, introduced in 1941 by G. M. Mirakjan in the paper [11].

These operators are intensive studied by J. Favard in 1944 in the paper [4] and O. Szász in 1950 in the paper [18].

Let $n \in \mathbb{N}$ and the operators $V_n : C_2([0,\infty)) \to C([0,\infty))$ defined for any function $f \in C_2([0,\infty))$ by

$$(V_n f)(x) = (1+x)^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x} \right)^k f\left(\frac{k}{n} \right) \tag{2.4}$$

for any $x \in [0,\infty)$.

The operators $(V_n)_{n\geq 1}$ are named Baskakov operators, introduced in 1957 by V. A. Baskakov in the paper [2].

In the paper [5], M. Ismail and C. P. May consider the operators $(R_n)_{n\geq 1}$.

For $n \in \mathbb{N}$, $R_n : C([0,\infty)) \to C([0,\infty))$ is defined for any function $f \in C([0,\infty))$ by

$$(R_n f)(x) = e^{-\frac{nx}{1+x}} \sum_{k=0}^{\infty} \frac{n(n+k)^{k-1}}{k!} \left(\frac{x}{1+x} \right)^k e^{-\frac{kx}{1+x}} f\left(\frac{k}{n} \right), \tag{2.5}$$

for any $x \in [0,\infty)$.

We consider $I \subset \mathbb{R}$, I an interval and we shall use the function sets: $B(I) = \{f | f : I \to \mathbb{R},\ f \text{ bounded on } I\}$, $C(I) = \{f | f : I \to \mathbb{R},\ f \text{ continuous on } I\}$ and $C_B(I) = B(I) \cap C(I)$. For any $x \in I$, let the function $\psi_x : I \to \mathbb{R}$, $\psi_x(t) = t - x$, for any $t \in I$.

If $I \subset \mathbb{R}$ is a given interval and $f \in B(I)$, then the first order modulus of smoothness of f is the function $\omega_1 : [0,\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$\omega_1(f;\delta) = \sup\left\{ |f(x') - f(x'')| : x',\, x' \in I,\ |x' - x''| \leq \delta \right\}. \tag{2.6}$$

Theorem 2.1. *Let $I \subset \mathbb{R}$ be an interval, $a \in I$, $n \in \mathbb{N}$ and the function $f : I \to \mathbb{R}$, f is n times derivable in a. According to Taylor's expansion theorem for the function f around a, we have*

$$f(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + (x-a)^n \mu(x-a) \tag{2.7}$$

where μ is a bounded function and $\lim\limits_{x\to a} \mu(x-a) = 0$.

If $f^{(n)}$ is a continuous function on I, then for any $\delta > 0$

$$|\mu(x-a)| \leq \frac{1}{n!} \left[1 + \delta^{-2}(x-a)^2 \right] \omega_1\left(f^{(n)}; \delta \right) \tag{2.8}$$

for any $x \in I$.

Proof. For the proof see [15]. $\square$

3. Main results

In the following, let s be a fixed natural number, s even.

Theorem 3.2. *Let $f : I(a,b) \to \mathbb{R}$ be a function, $f \in \mathcal{F}(p)$.*

a) *If $x \in I(a,b)$, f is a s times derivable function in x and the function $f^{(s)}$ is continuous in x, then*

$$\lim_{n\to\infty} n^{\frac{s}{2}} \left[(S_n f)(x) - \sum_{i=0}^{s} \frac{1}{n^i i!} A_{n,i}(x) f^{(i)}(x) \right] = 0. \tag{3.1}$$

b) *If f is a s times derivable function on $I(a,b)$, the function $f^{(s)}$ is continuous on $I(a,b)$, K is a compact set, $K \subset I(a,b)$, then the convergence given in (3.1) is uniform on K and there exists $k_s(K)$, $k_{s+2}(K) \in \mathbb{R}$, there exists $n_s \in \mathbb{N}$ such that*

$$n^{\frac{s}{2}} \left| (S_n f)(x) - \sum_{i=0}^{s} \frac{1}{n^i i!} A_{n,i}(x) f^{(i)}(x) \right| \leq \tag{3.2}$$
$$\leq \frac{1}{s!} \left[k_s(K) + k_{s+2}(K) \right] \omega_1 \left(f^{(s)}; \frac{1}{\sqrt{n}} \right)$$

for any $x \in K$ and for any $n \in \mathbb{N}$, $n \geq n_s$.

Proof. a) According to Taylor's theorem applied for the function f in the point x, we have

$$f(t) = \sum_{i=0}^{s} \frac{(t-x)^i}{i!} f^{(i)}(x) + (t-x)^s \mu(t-x) \tag{3.3}$$

for any $t \in I(a,b)$, where μ is a bounded function and $\lim\limits_{t\to x} \mu(t-x) = 0$.
In (3.3) multiplying by $W_n(x,t)$ and integrating on $I(a,b)$, we obtain

$$(S_n f)(x) = \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!} \int_a^b W_n(x,t)(t-x)^i dt + \int_a^b W_n(x,t)(t-x)^s \mu(t-x) dt.$$

Taking into account of the definition from the centered moment of m order, the relation above becomes

$$(S_n f)(x) = \sum_{i=0}^{s} \frac{1}{n^i i!} A_{n,i}(x) f^{(i)}(x) + \int_a^b W_n(x,t)(t-x)^s \mu(t-x) dt,$$

so

$$n^{\frac{s}{2}} \left[(S_n f)(x) - \sum_{i=0}^{s} \frac{1}{n^i i!} A_{n,i}(x) f^{(i)}(x) \right] = (R_n f)(x), \tag{3.4}$$

where

$$(R_n f)(x) = n^{\frac{s}{2}} \int_a^b W_n(x,t)(t-x)^s \mu(t-x) dt, \tag{3.5}$$

for any $n \in \mathbb{N}$.

Then

$$|(R_nf)(x)| \le n^{\frac{s}{2}} \int_a^b W_n(x,t)(t-x)^s \, |\mu(t-x)| \, dt$$

and taking Theorem 2.1 into account, for any $\delta > 0$ we have

$$|(R_nf)(x)| \le n^{\frac{s}{2}} \int_a^b W_n(x,t)(t-x)^s \frac{1}{s!} \left[1+\delta^{-2}(t-x)^2\right] \omega_1\left(f^{(s)};\delta\right) dt =$$

$$= \frac{1}{s!} n^{\frac{s}{2}} \left[\int_a^b W_n(x,t)(t-x)^s dt + \delta^{-2} \int_a^b W_n(x,t)(t-x)^{s+2} dt \right] \omega_1\left(f^{(s)};\delta\right).$$

Taking into account of the definition from the centered moment of m order, the relation above becomes

$$|(R_nf)(x)| \le \frac{1}{s!} \left[\frac{A_{n,s}(x)}{n^{\frac{s}{2}}} + \delta^{-2} \frac{A_{n,s+2}(x)}{n^{\frac{s}{2}+2}} \right] \omega_1\left(f^{(s)};\delta\right),$$

and considering $\delta = \dfrac{1}{\sqrt{n}}$, we obtain

$$|(R_nf)(x)| \le \frac{1}{s!} \left[\frac{A_{n,s}(x)}{n^{\frac{s}{2}}} + \frac{A_{n,s+2}(x)}{n^{\frac{s+2}{2}}} \right] \omega_1\left(f^{(s)};\frac{1}{\sqrt{n}}\right), \tag{3.6}$$

for any $n \in \mathbb{N}$.

Taking Consequence 1.1 into account and considering the fact that

$$\lim_{n\to\infty} \omega_1\left(f^{(s)};\frac{1}{\sqrt{n}}\right) = \omega_1\left(f^{(s)};0\right) = 0,$$

we have that

$$\lim_{n\to\infty} (R_nf)(x) = 0. \tag{3.7}$$

From (3.4) and (3.7), (3.1) follows.

b) Taking into account the relations (3.4), (3.6) and Consequence 1.4, it results that the relation (3.2) holds, from which it results that the convergence from (3.1) is uniform on K. □

Theorem 3.3. *Let $f : I(a,b) \to \mathbb{R}$ be a function, $f \in \mathcal{F}(p)$.*

a) *If $x \in I(a,b)$, f is a s times derivable function in x and the function $f^{(s)}$ is continuous in x, then*

$$\lim_{n\to\infty} \left[(S_nf)(x) - f(x)\right] = 0 \tag{3.8}$$

if $s = 0$,

$$\lim_{n\to\infty} n^{\frac{s}{2}} \left[(S_nf)(x) - \sum_{i=0}^{s-1} \frac{1}{n^i i!} A_{n,i}(x) f^{(i)}(x) \right] = \frac{f^{(s)}(x)}{s!} \, l_s(x) \tag{3.9}$$

and

$$\lim_{n\to\infty} n^{\frac{s}{2}}\left[(S_nf)(x)-\sum_{i=0}^{s-2}\frac{1}{n^i i!}A_{n,i}(x)f^{(i)}(x)\right]= \tag{3.10}$$
$$=\frac{f^{(s-1)}(x)}{(s-1)!}\,l_{s-1}(x)+\frac{f^{(s)}(x)}{s!}\,l_s(x)$$

if $s\geq 2$*, where* $l_{s-1}(x)$ *and* $l_s(x)$ *are defined in* (1.10). b) *If* f *is a* s *times derivable function on* $I(a,b)$*, the function* $f^{(s)}$ *is continuous on* $I(a,b)$ *and* K *is a compact set,* $K\subset I(a,b)$*, then the convergence from* $(3.8)-(3.10)$ *are uniform on* K*.*

Proof. It results from Theorem 3.1 and Consequence 1.1. □

Theorem 3.4. *Let* $f: I(a,b)\to\mathbb{R}$ *be a function,* $f\in\mathcal{F}(p)$*.*

a) *If* $x\in I(a,b)$*,* f *is a* s *times derivable in* x *and the function* $f^{(s)}$ *is continuous in* x*, then*

$$\lim_{n\to\infty}\left[(S_nf)(x)-f(x)\right]=0 \tag{3.11}$$

if $s=0$*,*

$$\lim_{n\to\infty} n\left[(S_nf)(x)-f(x)\right]=\frac{p(x)}{2}\,f''(x) \tag{3.12}$$

if $s=2$ *and*

$$\lim_{n\to\infty} n^2\left[(S_nf)(x)-f(x)-\frac{p(x)}{2n}\,f''(x)\right]= \tag{3.13}$$
$$=\frac{p'(x)p(x)}{6}\,f'''(x)+\frac{p^2(x)}{8}\,f^{(IV)}(x)$$

if $s=4$*.*

b) *If* f *is a* s *times derivable function on* $I(a,b)$*, the function* $f^{(s)}$ *is continuous on* $I(a,b)$ *and* K *is a compact set,* $K\subset I(a,b)$*, then the convergence from* $(3.11)-(3.13)$ *are uniform on* K*.*

Proof. It results from Theorem 3.2, Lemma 1.1, Lemma 1.2, Consequence 1.1, Consequence 1.2 and Consequence 1.4. □

In the following, by particularization and applying Theorem 3.2 or Theorem 3.3, we give Voronovskaja's type theorem for some known operators, for example the Bernstein operators, the Mirakjan-Favard-Szász operators, the Baskakov operators, the $(R_n)_{n\geq 1}$ operators, the Post Widder operators and the Gauss-Weierstrass operators. These operators are exponential operators (see [1], [9] or [17]).

Application 3.1. If $a=0$, $b=1$ and $p(x)=x(1-x)$, $x\in[0,1]$, we obtain the Bernstein operators. Because $C([0,1])\subset\mathcal{F}(p)$, Theorem 3.2 and Theorem 3.3 hold for any function $f\in C([0,1])$.

In 1932, E. Voronovskaja gave the relation (3.12) in the paper [19]. In the same year, S. Bernstein gave the relation (3.13) in the paper [3].

Application 3.2. If $a = 0$, $b = \infty$ and $p(x) = x$, $x \in [0, \infty)$, we obtain the Mirakjan-Favard-Szász operators. Theorem 3.2 and Theorem 3.3 hold for $C_2([0, \infty)) \cap \mathcal{F}(p)$ set functions.

Application 3.3. If $a = 0$, $b = \infty$ and $p(x) = x(1 + x)$, $x \in [0, \infty)$, we obtain the Baskakov operators. Theorem 3.2 and Theorem 3.3 hold for $C_2([0, \infty)) \cap \mathcal{F}(p)$ set functions.

Application 3.4. If $a = 0$, $b = \infty$ and $p(x) = x(1 + x)^2$, $x \in [0, \infty)$, we obtain the defined operator in (2.5). Theorem 3.2 and Theorem 3.3 hold for $C([0, \infty)) \cap \mathcal{F}(p)$.

Application 3.5. If $a = 0$, $b = \infty$ and $p(x) = x^2$, $x \in [0, \infty)$, we obtain the Post-Widder operators $(S_{1,n})_{n\geq 1}$, defined by

$$(S_{1,n}f)(x) = \frac{1}{(n-1)!} \left(\frac{n}{x}\right)^n \int_0^\infty e^{-\frac{nt}{x}} t^{n-1} f(t) dt \tag{3.14}$$

for any $f \in C([0, \infty))$, any $x \in [0, \infty)$ and any $n \in \mathbb{N}$. Theorem 3.2 and Theorem 3.3 hold for $C([0, \infty)) \cap \mathcal{F}(p)$ set function.

Application 3.6. If $a = -\infty$, $b = \infty$ and $p(x) = 1$, $x \in \mathbb{R}$, we obtain the Gauss-Weierstrass operators $(S_{2,n})_{n\geq 1}$, defined by

$$(S_{2,n}f)(x) = \sqrt{\frac{n}{2\pi}} \int_{\mathbb{R}} e^{-\frac{n(x-t)^2}{2}} f(t) dt \tag{3.15}$$

for any $f \in C(\mathbb{R})$, any $x \in \mathbb{R}$ and any $n \in \mathbb{N}$. Theorem 3.2 and Theorem 3.3 hold for $C(\mathbb{R}) \cap \mathcal{F}(p)$ set function.

Observation 3.1. We ask our selves if a relation of the same type as the one in Theorem 3.1 takes place for odd number s. The answer is negative. Considering in the following an function $f : I(a, b) \to \mathbb{R}$, $f \in \mathcal{F}(p)$, f is not a polynomial function with the degree that is mostly equal to 3, $x \in I(a, b)$ arbitrary, f is a s times derivable function in x and the function $f^{(s)}$ is continuous in x. Assuming the contrary, then for $s = 3$, there exists $\alpha_3 \in \mathbb{R}$ such that

$$\lim_{n\to\infty} n^{\alpha_3} \left[(S_n f)(x) - \sum_{i=0}^{3} \frac{1}{n^i i!} A_{n,i}(x) f^{(i)}(x)\right] = 0. \tag{3.16}$$

From Theorem 3.2 for $s = 4$, we have

$$\lim_{n\to\infty} n^2 \left[(S_n f)(x) - \sum_{i=0}^{3} \frac{1}{n^i i!} A_{n,i}(x) f^{(i)}(x)\right] = \frac{p^2(x)}{8} f^{(IV)}(x). \tag{3.17}$$

From (3.16) and (3.17), we obtain a contradiction, because $p(x)$ is strictly positive for $x \in I(a, b)$ and f is not a polynomial function with the degree that is mostly equal to 3.

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