CREATIVE MATH. & INF.
16 (2007), 70 - 74

Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

A direct finding of the supremum of sequences explained by a fixed point theorem and some new results in asymptotic analysis

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ABSTRACT. We present an explanation by a fixed point theorem of a situation in the theory of the sequences. Some additional facts are given.

1. INTRODUCTION

The theory of the fixed point has important classical and old applications especially in Analysis. It would be naive to try to make here a complete list of these applications. Then we will cite only some of these. So, the theorem of Knaster permits a definition of the root of order n of a real, positive number (see [4]). The Banach's contraction principle (also called of Banach-Caccioppoli) based the Picard iteration, gives the solution in the theory of the implicit functions, for the initial value problem for ordinary differential equations, for some integral equations and some integro-differential equations by using the Picard iteration (see [4], [5]). Recently Professor *I. A. Rus* and some of his collaborators has obtained in [6], [7], [1], [2], a series of results concerning the iterates of the approximation operators of Bernstein and Stancu type, via Picard iteration and the contraction principle.

In this work we intend to explain a particular situation on the real axis, namely of the theory of the sequences of real numbers, using the fixed point theory.

2. POSING THE PROBLEM

Let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence of real numbers and let $A = \{a_n \mid n \in \mathbb{N}\}$. By a well-known theorem, if $(a_n)_n$ has an upper bound, it is convergent and $\lim\limits_{n\to\infty} a_n = \sup A$.

Usually, when we study if a sequence has an upper bound, we obviously do not search for $\sup A$; so we can consider some classical examples as the following:

$a_n = \left(1+\dfrac{1}{n}\right)^n < 3$ (when $\sup A = \mathrm{e} = 2,71828\ldots$, the famous constant of Napier)

$a_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \ldots + \dfrac{1}{2n} < 1$ (when $\sup A = \ln 2 = 0,69314\ldots$)

$a_n = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \ldots + \dfrac{1}{n^2} < 2$ (when $\sup A = \dfrac{\pi^2}{6} = 1,64493\ldots$)

Received: 9.09.2006. In revised form: 12.01.2007.

2000 *Mathematics Subject Classification.* 26D15, 30B10, 33F05.

Key words and phrases. Sequence, order of convergence, set, upper bound, lower bound, supremum, infimum, fixed point of a function.

$a_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n+1) < 1$ (when $\sup A = \gamma = 0,57721\ldots$, the constant of Euler, also called the constant of Euler-Mascheroni).

But there exist certain sequences for which, searching for a majorant, we obtain directly $\sup A$.

A typical example is the following: consider the sequence $(a_n)_n$ defined by the equalities $a_1 = \sqrt{\alpha}$ and $a_{n+1} = \sqrt{\alpha + a_n}$, for $n \geq 1$, where $\alpha \geq 0$ is a given constant. The sequence is strictly increasing and the majorant is exactly $(1 + \sqrt{1 + 4\alpha})/2$.

3. The main result

The above situation represents a particular case of a theorem which we will present now. These situations are generally known; we present here a precise form (which also admits a dual form).

Theorem 1. *Let I be an interval of $\mathbb{R}$ and let $f : I \to I$ be a function which has the following properties*:

(i) *f is strictly increasing*; (ii) *f has a fixed point $x^* \in I$*; (iii) *$x < f(x)$, for any $x \in I$, $x < x^*$.*

Then we have:

(a) *the point x^* is the unique fixed point of the function f on the interval $I \cap (-\infty, x^*]$*;

(b) *f is left continuous at x^**;

(c) *the sequence $(a_n)_n$ defined by the recurrence relation $a_{n+1} = f(a_n), n = 1, 2, 3, \ldots$, with $a_1 \in I \cap (-\infty, x^*)$ given, has the following properties*:

(α) *it is strictly increasing*;

(β) *it is upper-bounded by x^** (*therefore $(a_n)_n$ is convergent*);

(γ) $\lim_{n\to\infty} a_n = x^*$ (*therefore $x^* = \sup A$*).

(So, under the assumptions above, we obtain à priori that an upper bound for the sequence $(a_n)_n$ is $\sup A$).

Proof. (a) Suppose, ab absurdum, that the function f has also another fixed point $x^{**} \in I \cap (-\infty, x^*]$, $x^{**} \neq x^*$ i.e. $x^{**} = f(x^{**})$. So, we have $x^{**} < x^*$. But, from the hypothesis (iii) it results $x^{**} < f(x^{**})$ that is a contradiction. Therefore the fixed point x^* is unique.

(b) Let $(x_n)_n$ be an arbitrary sequence of real numbers in $I \cap (-\infty, x^*)$,

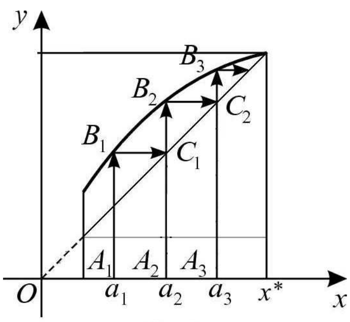

Fig. 1

which tends to x^*: $\lim_{n\to\infty} x_n = x^*$.

So, from (iii), we have:

$$x_n < f(x_n). \tag{1}$$

By the other hand, from the inequality $x_n < x^*$, it results $f(x_n) < f(x^*)$, i. e. (x^* being a fixed point):

$$f(x_n) < x^*. \tag{2}$$

Therefore, we have:

$$x_n < f(x_n) < x^*. \tag{3}$$

The sequence $(x_n)_n$ from the left part converges to x^*, so we obtain for $n \to \infty$:

$$\lim_{\substack{n\to\infty \\ (x_n \nearrow x^*)}} f(x_n) = x^*.$$

But x^* is a fixed point of f, therefore the last relation becomes:

$$\lim_{\substack{n\to\infty \\ (x_n \nearrow x*)}} f(x_n) = f(x^*).$$

Therefore f is left-continuous at x^*.

(c) (α) We have, from (iii), $a_1 < f(a_1) = a_2$, i. e. $a_1 < a_2$. Suppose now that, for a certain n, we have $a_n < a_{n+1}$. Because of the fact that f is strictly increasing, it results $f(a_n) < f(a_{n+1})$, that is $a_{n+1} < a_{n+2}$. So, it follows by induction, that the sequence $(a_n)_n$ is strictly increasing.

(β) We have $a_1 < x^*$. Suppose now that, for a certain n, we have $a_n < x^*$. It results $f(a_n) < f(x^*)$, that is $a_{n+1} < x^*$. So, we get by induction, that $a_n < x^*$ for any $n \in \mathbb{N}$.

(γ) Let $l = \lim_{n\to\infty} a_n$. It results that we have also $\lim_{n\to\infty} a_{n+1} = l$. From the inequality $a_n < x^*$, for any $n \in \mathbb{N}$, it results $l \le x^*$.

The sequence $(a_n)_n$ was defined by the recurrence relation $a_{n+1} = f(a_n)$. Passing to the limit for $n \to \infty$, we obtain:

$$\lim_{n\to\infty} a_{n+1} = \lim_{\substack{n\to\infty \\ (a_n \nearrow x^*)}} f(a_n).$$

But f is left-continuous at the point x^*, therefore f commutes with the left limit and the last equality gives us:

$$\lim_{n\to\infty} a_{n+1} = f\left(\lim_{\substack{n\to\infty \\ (a_n \nearrow x^*)}} a_n\right).$$

i. e.

$$l = f(l),$$

where $l \in I \cap (-\infty, x^*]$, which means that l is a fixed point of the function f on the interval $I \cap (-\infty, x^*]$. But f has an unique fixed point in this interval, namely x^*. It results that $l = x^*$. □

So, this fixed point theorem gives us an explanation of the fact mentioned in Section 2, regarding the recurrent sequence defined by the equalities $a_1 = \sqrt{\alpha}$, $a_{n+1} = \sqrt{\alpha + a_n}$, where $\alpha \ge 0$. As we have spoken, the a priori upper bound is just the fixed point of the associated function

$$f : [0,\infty) \to [0,\infty),\ f(x) = \sqrt{\alpha + x}.$$

Many other similar examples can be considered, e. g. the sequence defined by the equalities:

$$a_{n+1} = \sqrt[3]{(a_n^3 + 3a_n + 4)/2}, \quad a_1 = 1.$$

The scheme is completely similar (see Figure 2).

The limit has a more „complicated" form being a so called cubic irrational number, namely $x^* = \sup A = \sqrt[3]{2+\sqrt{3}} + \sqrt[3]{2-\sqrt{3}}$, which is the fixed point of f. It is the unique real root of the equation $f(x) = x$, which becomes $x^3 - 3x - 4 = 0$ (this gives us the form of cubic irrational number of x^*).

This is an a priori upper bound for $(a_n)_n$, given by $\sup A$.

A more elementary and detailed exposition of this subject is given in our previous work [8].

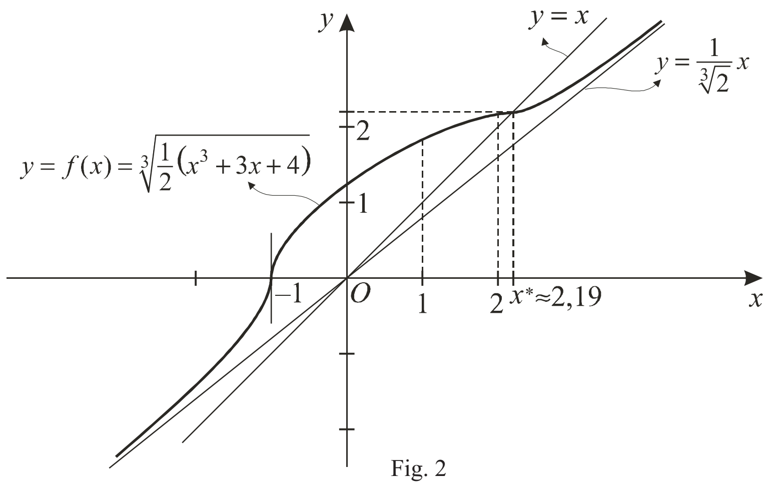

Fig. 2

4. The dual theorem

Of course, a similar situation is valid for the dual relation in which an a priori lower bound of $(a_n)_n$ is directly obtained as $\inf A$. We have the following

Theorem 2. *Let I be an interval of $\mathbb{R}$ and let $f : I \to I$ be a function which has the following properties*:

(i) *f is strictly increasing*;

(ii) *f has a fixed point $x^* \in I$*;

(iii) *$f(x) < x$, for any $x \in I$, $x > x^*$.*

Then we have:

(a) *the point x^* is the unique fixed point of the function f on the interval $I \cap [x^*, \infty)$*;

(b) *f is right continuous at x^**;

(c) *the sequence $(a_n)_n$ defined by the recurrence relation $a_{n+1} = f(a_n)$, $n = 1, 2, 3, \ldots$ with $a_1 \in I \cap (x^*, \infty)$ given, has the following properties*:

(α) *it is strictly decreasing*;

(β) *it is lower bounded by x^* (therefore $(a_n)_n$ is convergent)*;

(γ) $\lim_{n\to\infty} a_n = x^*$ (*therefore* $x^* = \inf A$).

(So, under the assumption above, we obtain a priori that an upper bound for the sequence $(a_n)_n$ is $\inf A$.)

5. Order of convergence

Consider again the sequence defined by the equations $a_1 = \sqrt{\alpha}$ and $a_{n+1} = \sqrt{\alpha + a_n}$, $n = 1, 2, 3, \ldots$, where $\alpha \geq 0$ is a given number, let $\alpha \geq 1$. The limit of this sequence was obtained by an iteration of Picard-Banach type, $a_{n+1} = f(a_n)$, related to the function $f : [0, \infty) \to [0, \infty)$, $f(x) = \sqrt{\alpha + x}$. Because of the relation:

$$f'(x) = \frac{1}{2\sqrt{\alpha + x}} \leq \frac{1}{2\sqrt{\alpha}} < \frac{1}{2},$$

the function f is a contraction of constant $k = 1/2$. But the Banach contraction principle assures us that if f is a k-contraction of a complete metric space $(X; d)$, $x_1 \in X$ is given, $x_{n+1} = f(x_n)$, then f has an unique fixed point, namely $x = \lim\limits_{n\to\infty} x_n$ and we have:

$$d(x_n; x) \leq \frac{k^{n-1}}{1-k} d(x_1, x_2).$$

In our case, this result becomes:

$$|a_n - l| \leq \frac{1}{2^{n-2}} \left(\sqrt{\alpha + \sqrt{\alpha}} - \sqrt{\alpha} \right).$$

If $\alpha = 1$, then $(a_n)_n$ is the sequence of general term

$$a_n = \sqrt{1 + \sqrt{1 + \ldots + \sqrt{1}}}$$

(with n square roots), which converges to the "golden number" $\varphi = (1 + \sqrt{5})/2$. So we obtain:

$$|a_n - \varphi| < \frac{1}{2^{n-2}}(\sqrt{2} - 1).$$

Aknowledgment. I thank Professor Dr. *Ioan A. Rus* for his kind explanations concerning the finding of the iterates of the approximation operators using fixed point methods. I also thank Professor Dr. *Vasile Berinde* for some valuable comments.

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