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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

The study of some nonlinear dynamical systems modelled by a more general Rayleigh-Van Der Pol equation

MIRCEA LUPU and FLORIN ISAIA

ABSTRACT. In this paper, we study the mathematical model for nonlinear dynamical systems with distributed parameters given by a generalized Rayleigh-Van der Pol equation. In the autonomous case and in the non-autonomous case, conditions of stability, bifurcations, self-oscillations are studied using criteria of Liapunov, Bendixon, Hopf [11], [12]. Asymptotic and numerical methods are often used [5]. The equation has the form

$$\ddot{x} + \omega^2 x = \left(\alpha - \beta x^2 - \gamma \dot{x}^2\right) \dot{x} + f\left(t\right),$$

where resonance and limit cycles can be remarked [1]. Note that for $\beta = 0$, $\alpha \neq 0$, $\gamma \neq 0$ we have the Rayleigh equation [1], while for $\gamma = 0$, $\alpha \neq 0$, $\beta \neq 0$ we have the Van der Pol equation [2],[3]. Besides the theoretical study, the applications to techniques are very important: dynamical systems in the mechanics of vibrations, oscillations in electromagnetism and transistorized circuits [6], the aerodynamics of the flutter with two degrees of freedom [8], are modelled by this hybrid equation proposed by authors.

1. INTRODUCTION

In this paper we will study nonlinear equations of the type

$$\ddot{x} + f\left(x, \dot{x}\right) \dot{x} + g\left(x\right) = F\left(t\right), \tag{1.1}$$

where, if $f\left(x, \dot{x}\right)$ is a polynomial, an equation of Rayleigh-Van der Pol (RVP) form can be obtained

$$m\ddot{x} + c\dot{x} + kx = \left(A - Bx^2\right) \dot{x} - C\dot{x}^3 + Dsin\nu t, \tag{1.2}$$

$$\ddot{x} + \omega^2 x = \left(\alpha - \beta x^2 - \gamma \dot{x}^2\right) \dot{x} + f\left(t\right). \tag{1.2'}$$

Thus, for $\beta = 0$, $\alpha \neq 0$, $\gamma \neq 0$ we have equations of Rayleigh type [1], while for $\gamma = 0$, $\alpha \neq 0$, $\beta \neq 0$ we have equations of Van der Pol type [2], [3]. They represent mathematical models for phenomena in mechanic vibrations, fluid oscillations [5], electrical transistorized circuits, generating lamps [11], flutter oscillations in aerodynamics [8], dampers with friction [10], astronomical phenomena, etc. The forces $f\left(t\right)$ can be harmonic (non-autonomous case for $f \neq 0$, autonomous case for $f = 0$).

Some studies on the stability of solutions, on the bifurcations and on the resonance can be made directly, using the distributed parameters α, β, γ, ω or using small parameter methods, asymptotic methods, numeric methods. Here is a technique to specify a small parameter μ on equation (1.2): dividing by m, denoting

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$c/m = 2n$, $k/m = \omega^2$, $2n/\nu = \mu$, $\left(\omega^2 - \nu^2\right)/\nu^2 = \mu\chi$ and making the change of variable $\tau = \nu t$, we have $\dot{x} = x'(t)\,\nu$, $\ddot{x} = x''(t)\,\nu^2$ and equation (1.2) becomes

$$x'' + x = \mu\left[-\chi x + \left(\alpha - \beta x^2\right)x' - \gamma {x'}^3 + f_0 \sin\tau\right]. \tag{1.3}$$

For equation (1.2′) in the autonomous case ($f = 0$), we can consider $\tau = \omega t$, $\dot{x} = x'(t)\,\omega$, $\ddot{x} = x''(t)\,\omega^2$, $\nu = \alpha/\omega$, $\varepsilon = \beta/\alpha$, $\delta = \alpha\gamma/\nu^2$ and equation (1.2′) becomes

$$x'' + x = \mu x'\left(1 - \varepsilon x^2 - \delta {x'}^2\right). \tag{1.4}$$

In the autonomous case and in the non-autonomous case, we will study the stability of solutions, bifurcations, self-oscillations and resonance.

2. The study of the (RVP) equation in the autonomous case

In the phase space (x, y), equation (1.2′) with $f = 0$ becomes

$$\dot{x} = y, \quad \dot{y} = -\omega^2 x + \left(\alpha - \beta x^2 - \gamma y^2\right)y, \tag{2.5}$$

and is a special case of the systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y). \tag{2.6}$$

The equilibrium points are the solutions of the algebraic system with unknowns (x, y) [4]

$$P(x, y) = 0, \quad Q(x, y) = 0.$$

We will study the stability of the equilibrium point $O^*\,(x^* = 0, y^* = 0)$. The system (2.5) being nonlinear, we consider the linear system in first approximation corresponding to (2.5)

$$\dot{x} = y, \quad \dot{y} = -\omega^2 x + \alpha y, \tag{2.7}$$

and we will study this system in a neighbourhood of $O^*\,(0, 0)$. The characteristic equation is

$$\begin{vmatrix} \frac{\partial P}{\partial x}(x^*, y^*) - r & \frac{\partial P}{\partial y}(x^*, y^*) \\ \frac{\partial Q}{\partial x}(x^*, y^*) & \frac{\partial Q}{\partial y}(x^*, y^*) - r \end{vmatrix} = 0 \Leftrightarrow r^2 - \alpha r + \omega^2 = 0, \tag{2.8}$$

with the corresponding roots

$$r_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 4\omega^2}}{2}.$$

If $\alpha < 0$, then O^* is an asymptotic stable point for the linear system (2.7) as for the nonlinear system (2.5). In fact, for $\alpha \in (-\infty, -2\omega]$, O^* is an asymptotic stable node and for $\alpha \in (-2\omega, 0)$, O^* is an asymptotic stable focus [2], [4].

If $\alpha > 0$, then O^* is an unstable point for the linear system (2.7) as for the nonlinear system (2.5), the trajectories leaving O^* when $t \to \infty$. Using some theorems of Liapunov and Bendixon type, we will show the existence of a stable or unstable Hopf bifurcation in the neighbourhood of $\alpha = \alpha_0 = 0$ and we will specify the limit cycle for the trajectories of system (2.5). In order to verify the non-existence of periodic trajectories which can be limit cycles, we will use the Bendixon criterion [1], [11]: *The system (2.6) has a periodic solution only if the expression $\partial P/\partial x + \partial Q/\partial y$ changes its sign or takes zero value.* Here, $\partial P/\partial x + \partial Q/\partial y = \alpha - \beta x^2 - 3\gamma y^2$. Therefore, for $\alpha > 0$, O^* is an unstable point, but a closed curve which intersects the ellipse of equation $\beta x^2 + 3\gamma y^2 - \alpha = 0$ can exists. If this curve is a stable limit cycle, then it contains the unstable point O^*. In this case, we have a Hopf bifurcation [6]: *If the characteristic polynomial (2.8) has complex roots of the type* $r_{1,2} = \Lambda(\alpha) \pm i\Omega(\alpha)$, *with* $\Lambda(\alpha_0) = 0$, $\Lambda(\alpha) < 0$ *for* $\alpha < \alpha_0$, $\Lambda(\alpha) > 0$ *for* $\alpha > \alpha_0$, $(\partial\Lambda/\partial\alpha)\,|_{\alpha=\alpha_0} > 0$ *and* $\Omega(\alpha_0) = \Omega_0 \neq 0$, *then for* $\alpha > \alpha_0$ *sufficiently small, the system (2.6) admits a periodic solution.* Here, $\alpha_0 = 0$, $\Lambda(\alpha) = \alpha/2$, $\Omega(\alpha) = \sqrt{4\omega^2 - \alpha^2}/2$ and hence, for $\alpha > 0$ the system (2.5) admits a periodic solution which is a limit cycle.

In order to specify the stability of the limit cycle, we will use the complex form of system (2.5). We have $z = x + iy$, $\bar{z} = x - iy$, $\dot{x} = \dot{x} + i\dot{y}$. By adding the two equations of (2.5) we obtain

$$\dot{z} = A(\alpha)\, z + B(\alpha)\, \bar{z} + g(z, \bar{z}),$$

where $g(z, \bar{z})$ contains monomials of degrees 2 and 3 in $(z, \bar{z})$ [6], [8]

$$g(z, \bar{z}) = \frac{g_{20}}{2} z^2 + g_{11} z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 + \frac{g_{21}}{2} z^2\bar{z} + \frac{g_{12}}{2} z\bar{z}^2 + \frac{g_{30}}{6} z^3 + \frac{g_{03}}{6}\bar{z}^3 + O\left(|z|^4\right).$$

Define the first Liapunov coefficient to be the real number

$$l_1(\alpha_0) = \frac{\operatorname{Re} c_1(\alpha_0)}{\Omega_0}, \text{ with } c_1(\alpha_0) = \frac{i}{2\Omega_0}\left(g_{20} g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}.$$

It is also known that $\operatorname{sgn} l_1(\alpha_0) = \operatorname{sgn} \operatorname{Re}(i g_{20} g_{11} + \Omega_0 g_{21})$. Now, we state the topological Hopf criterion:

- *if* $l_1(\alpha_0) < 0$, *then the periodic solution is a stable limit cycle with supercritical Hopf bifurcation;*
- *if* $l_1(\alpha_0) > 0$, *then the periodic solution is an unstable limit cycle with subcritical Hopf bifurcation.*

In the case of system (2.5), we have the complex form

$$\begin{aligned}\dot{z} \;=\;& \frac{\alpha - i\left(1+\omega^2\right)}{2} z - \frac{\alpha + i\left(1+\omega^2\right)}{2}\bar{z} \\ &+ \frac{\gamma - \beta}{8} z^3 - \frac{3\gamma + \beta}{8} z^2\bar{z} + \frac{3\gamma + \beta}{8} z\bar{z}^2 - \frac{\gamma - \beta}{8}\bar{z}^3,\end{aligned}$$

with $\frac{g_{21}}{2} = -\frac{3\gamma+\beta}{8}$. For $\alpha > 0$, we have $\operatorname{sgn} l_1(\alpha_0) = \operatorname{sgn} \operatorname{Re}\left(-\Omega_0 \frac{3\gamma+\beta}{4}\right) = -1$. Therefore, the periodic solution is a stable limit cycle which contains the unstable point O^*.

The same conclusions can be derived by applying the variation of constants method (Van der Pol). We start from the equation

$$x'' + x = \mu x'\left(1 - \varepsilon x^2 - \delta x'^2\right) \tag{1.4}$$

which is obtained from (1.2′) by using the following transformations (see Section 1): $\tau = \omega t$, $\dot{x} = x'(\tau)\,\omega$, $\ddot{x} = x''(\tau)\,\omega^2$, $\mu = \alpha/\omega$, $\varepsilon = \beta/\alpha$, $\delta = \alpha\gamma/\mu^2$. Here, μ is a small parameter. For $\mu = 0$ ($\alpha = 0$) we have the solution $x = a\cos\tau + b\sin\tau$, $a, b \in \mathbb{R}$. In the following, we will consider that $a = a(\tau)$, $b = b(\tau)$ are parameters with slow variation and by using the variation of constants method, we have

$$a'\cos\tau + b'\sin\tau = 0, \tag{2.9}$$

$$x' = -a\sin\tau + b\cos\tau, \quad x'' = -a'\sin\tau + b'\cos\tau - a\cos\tau - b\sin\tau. \tag{2.10}$$

By replacing x, x', x'' in equation (1.4) and denoting $f(x, x') = x'\left(1 - \varepsilon x^2 - \delta x'^2\right)$, we get

$$a' \sin\tau + b' \cos\tau = \mu f(a\cos\tau + b\sin\tau, -a\sin\tau + b\cos\tau). \tag{2.11}$$

The solution of system (2.9), (2.11) with unknowns (a', b') is

$$-a' = -\mu f \cdot \sin\tau, \quad b' = \mu f \cdot \cos\tau. \tag{2.12}$$

By averaging (2.12) with respect to the period $T = 2\pi$, we obtain

$$\begin{cases} a' = -\frac{\mu}{2\pi}\int_0^{2\pi} f(a\cos\tau + b\sin\tau, -a\sin\tau + b\cos\tau)\sin\tau d\tau, \\ b' = \frac{\mu}{2\pi}\int_0^{2\pi} f(a\cos\tau + b\sin\tau, -a\sin\tau + b\cos\tau)\cos\tau d\tau. \end{cases}$$

Here, we prefer a different variant of the variation of constants method. We consider the polar coordinates

$$a = A\cos\theta, \quad b = A\sin\theta, \tag{2.13}$$

where $A = A(\tau)$, $\theta = \theta(\tau)$ have a slow variation and we get

$$a' = A'\cos\theta - A\theta'\sin\theta, \quad b' = A'\sin\theta + A\theta'\cos\theta. \tag{2.14}$$

Therefore, the solution (x, y) with $y = \dot{x}$ becomes

$$x = A\cos(\tau - \theta), \quad y = \dot{x} = -A\omega\sin(\tau - \theta). \tag{2.15}$$

We have

$$A'\cos(\tau - \theta) + A\theta'\sin(\tau - \theta) = 0, \tag{2.16}$$

$$\begin{cases} x' = -A\sin(\tau - \theta), \\ x'' = -A'\sin(\tau - \theta) - A\cos(\tau - \theta) + A\theta'\cos(\tau - \theta). \end{cases} \tag{2.17}$$

Replacing (2.15), (2.17) in equation (1.4) we get

$$\begin{aligned} -A'\sin(\tau - \theta) + A\theta'\cos(\tau - \theta) &= -\mu A\sin(\tau - \theta) \\ &\cdot \left[1 - \varepsilon A^2\cos^2(\tau - \theta) - \delta A^2\sin^2(\tau - \theta)\right]. \end{aligned} \tag{2.18}$$

Now, we solve the system (2.16), (2.18) with unknowns (A', θ'), we average the results with respect to $T = 2\pi$ and then we perform the change of variable $\varphi = \tau - \theta = \omega t - \theta$. We obtain

$$\begin{cases} A' = \frac{\mu}{2\pi}\int_0^{2\pi} A\left(1 - \varepsilon A^2\cos^2\varphi - \delta A^2\sin^2\varphi\right)\sin^2\varphi d\varphi, \\ \theta' = -\frac{\mu}{2\pi}\int_0^{2\pi}\left(1 - \varepsilon A^2\cos^2\varphi - \delta A^2\sin^2\varphi\right)\sin^2\varphi d\varphi. \end{cases} \tag{2.19}$$

Taking account of $\dot{A} = A'\omega$, $\dot{\theta} = \theta'\omega$, relations (2.19) become

$$\begin{cases} \dot{A} = \frac{\mu\omega}{2\pi}\int_0^{2\pi} A\left(1 - \varepsilon A^2\cos^2\varphi - \delta A^2\sin^2\varphi\right)\sin^2\varphi d\varphi = \frac{\mu\omega}{2\pi}\Phi(A), \\ \dot{\theta} = -\frac{\mu\omega}{2\pi}\int_0^{2\pi}\left(1 - \varepsilon A^2\cos^2\varphi - \delta A^2\sin^2\varphi\right)\sin^2\varphi d\varphi = -\frac{\mu\omega}{2\pi}\Psi(A). \end{cases} \tag{2.20}$$

The differential system (2.20) and the initial conditions $A(t_0 = 0) = A_0$, $\theta(t_0 = 0) = \theta_0$ allow us to find the amplitude $A(t)$ and the phase difference $\theta(t)$. Because $A(t)$ and $\theta(t)$ have a slow variation, the algebraic system $\Phi(A) = 0$, $\Psi(A) = 0$ allows us to find the critical points $A = A^* = \text{const}$. We have

$$\theta = 0,\ A = \frac{A^*}{\sqrt{1 - Ce^{-\mu\omega t}}}, \quad \text{with } A^* = \frac{2}{\sqrt{\varepsilon + 3\delta}},\ C = 1 - \left(\frac{A^*}{A_0}\right)^2. \tag{2.21}$$

Consequently, from (2.15) and (2.21) we have the solution

$$x = A(t)\cos\omega t, \quad y = -A(t)\,\omega\sin\omega t, \tag{2.22}$$

with $A(t)$ given by (2.21). Relation (2.21) shows us that in the case $\alpha > 0$ ($\mu > 0$) and $\beta > \gamma > 0$, the trajectories are spiral lines which tend (when $t \to \infty$) to the stable limit cycle represented by the ellipse of equation $x^2 + y^2/\omega^2 = A^{*2}$.

Remark 2.1. If $\gamma = 0$, $\alpha \neq 0$, $\beta \neq 0$, we obtain the special Van der Pol case studied by us in [9]. If $\beta = 0$, $\alpha \neq 0$, $\gamma \neq 0$, we obtain the special Rayleigh case studied by us in [8] for the aerodynamics of the flutter with two degrees of freedom. It is worth to point out the studies made by N. Mureşan, N. Vornicescu, and N. Lungu [7] concerning the boundness of solutions and the calculus of Liapunov functions for these types of nonlinear equations.

SELF-OSCILLATIONS. They are oscillations produced by the energy transmitted to a system from sources with non-oscillating character: damping or resistance with dry friction, wind, maintained self-induction. Inside of motion equations, damping forces with variable coefficients appear and in certain moments, the characteristic equation admits real positive roots (instead of complex roots) which increase the amplitude. The phenomenon of resonance appears when we have perturbing external forces, while self-oscillations appear because of an interaction between internal forces, particularly if these forces depend on speed with high exponents. For example, we have self-oscillations in equations like:

$$\ddot{x} - \left(c_1 - c_2 x^2\right)\dot{x} + kx = 0,$$
$$\ddot{x} - \left(c_1 - c_2 \dot{x}^2\right)\dot{x} + kx = 0,$$

if the conditions $|x| < \sqrt{c_1/c_2}$, respectively $|\dot{x}| < \sqrt{c_1/c_2}$, are fulfilled for a short period of time. After this period, the amplitude $A(t)$ satisfy $A(t) \to A^*$ when $t \to \infty$, independently of the initial conditions.

Returning to (2.20), we remark that we have to find $A = A(t, A_0)$ and we have to determine a limit stationary amplitude A^*, when $t \to \infty$. The motivation of this operation is that the equation $\dot{A} = [(\mu\omega)/(2\pi)]\,\Phi(A)$ represents the variation of the energy and of the mechanical work produced by the nonlinear terms $f(x, \dot{x})$ of an equation of the form $\ddot{x} + x = f(x, \dot{x})$. If the mechanical work $f(x, \dot{x})\,dx = f(x, \dot{x})\,\dot{x}dt$ is negative for a short period of time, then the energy decreases and the motion is amortized. If the mechanical work is positive for a short period of time, then the energy decreases. It is easy to see the part of $\Phi(A)$ in the calculation of the mechanical work:

$$\int_0^{2\pi} f dx = \pi A\left(1 - \frac{A^2(\varepsilon + 3\delta)}{4}\right) = \Phi(A)$$

These two situations do not assure an energetic balance for all periods, therefore the condition for the uniformity of self-oscillations is

$$\Phi(A) = \int_0^{2\pi/\omega} f(A\cos\omega t, -A\omega\sin\omega t)\sin\omega t dt = 0.$$

We find A^* which verifies $\Phi(A^*) = 0$ and $A(t) \to A^*$ when $t \to \infty$. For $\alpha > 0$, the critical point is $A^* = \sqrt{4/(\varepsilon + 3\delta)} = \sqrt{4\alpha/(\beta + 3\gamma\omega^2)}$ and coincides with A^*

given by (2.21). By (2.21) we have

$$\lim_{t\to\infty} A(t) = A^* = \sqrt{\frac{4\alpha}{\beta+3\gamma\omega^2}}$$

and from (2.22) it follows that the trajectories from the first periods give self-oscillations with amplitude $A(t)$ which tend to A^* when $t\to\infty$ irrespectively of the initial conditions. These trajectories are spiral lines which tend to the stable limit cycle $x^2+y^2/\omega^2 = A^{*2}$. Observe that $x(t)\to\pm A^*$ and the asymptotes $x=\pm A^*$ to the graph $x=x(t)$ have initially a number of oscillations.

3. The study of the (RVP) system in the non-autonomous case

Consider the equation

$$\ddot{x}+\omega^2 x = \left(\alpha-\beta x^2-\gamma\dot{x}^2\right)\dot{x}+F_0\sin\nu t. \tag{3.23}$$

In Section 1 we presented a manner to specify a small parameter μ for equation (3.23). Making the substitutions $\nu t=\tau$, $\left(\omega^2-\nu^2\right)/\nu^2=\mu\chi$, $\lambda=\alpha/(\nu\mu)$, $\varepsilon=\beta/(\nu\mu)$, $\delta=\gamma\nu/\mu$, $f_0=F_0/\left(\nu^2\mu\right)$ we have $\dot{x}=x'(\tau)\nu$, $\ddot{x}=x''(\tau)\nu^2$ and equation (3.23) becomes

$$x''+x=\mu\left[-\chi x+\left(\lambda-\varepsilon x^2-\delta {x'}^2\right)x'+f_0\sin\tau\right]. \tag{3.24}$$

Thus, we can consider that equation (3.23) has the form

$$\ddot{x}+\omega^2 x=\mu\left(\alpha-\beta x^2-\gamma\dot{x}^2\right)\dot{x}+\mu f\sin\nu t. \tag{3.25}$$

In order to determine the solution $x=x(t)$ which satisfies the initial conditions $A(0)=A_0$, $\theta(0)=0$, we use the numeric method of Krilov-Bogoliubov-Mitropolski [2], [3], [5]: we seek for an asymptotic solution

$$x=A\cos(\omega t-\theta)+\mu x_1+\mu^2 x_2+\ldots+\mu^N x_N,$$

with $A(t)$, $\theta(t)$ with slow variation and $x_i=x_i(t)$. In the sequel, only the first approximation will be used. Hence

$$x=A\cos(\omega t-\theta)+\mu x_1. \tag{3.26}$$

We replace (3.26) in (3.25), we make the identification with respect to $\cos(\omega t-\theta)$, $\sin(\omega t-\theta)$ and then with respect to μ^1. In the obtained equations, we consider to be zero the terms from the right side containing $\dot{A}$, $\dot{\theta}$, so that we get the following equations

$$\begin{aligned} \ddot{A}+2A\omega\dot{\theta}-A\dot{\theta}^2 &= 0, && (3.27)\\ A\ddot{\theta}+2\dot{A}\dot{\theta}-2\dot{A}\omega &= \mu\left(-\alpha A\omega+\frac{1}{4}\beta\omega A^3+\frac{3}{4}\gamma\omega^3A^3\right), && (3.28)\\ \ddot{x}_1+\omega^2x_1 &= \frac{\omega A^3\left(\beta-\gamma\omega^2\right)}{4}\sin 3(\omega t-\theta)+f\sin\nu t. && (3.29) \end{aligned}$$

Equations (3.27), (3.28) are *the variational equations*, while equation (3.29) is *the perturbational equation*. Now, we replace equations (3.27), (3.28) with the following equations

$$\begin{aligned}\dot{\theta} &= 0,\\ -2\dot{A}\omega &= \mu\left(-\alpha A\omega + \frac{1}{4}\beta\omega A^3 + \frac{3}{4}\gamma\omega^3 A^3\right),\end{aligned}$$

which are equivalent to

$$\dot{\theta} = 0, \tag{3.30}$$

$$\dot{A} = \frac{\mu\alpha A\left(A^{*3} - A^2\right)}{2A^{*2}}, \quad \text{with } A^* = \sqrt{\frac{4\alpha}{\beta + 3\gamma\omega^2}}. \tag{3.31}$$

By integrating the differential system (3.30), (3.31) and by using the initial conditions $A(0) = A_0$, $\theta(0) = 0$, we obtain

$$\theta = 0,\ A = \frac{A^*}{\sqrt{1 - Ce^{-\mu\alpha t}}}, \quad \text{with } A^* = \sqrt{\frac{4\alpha}{\beta + 3\gamma\omega^2}},\ C = 1 - \left(\frac{A^*}{A_0}\right)^2. \tag{3.32}$$

Taking account of (3.32), equation (3.29) becomes

$$\ddot{x}_1 + \omega^2 x_1 = \frac{\omega A^3\left(\beta - \gamma\omega^2\right)}{4}\sin 3\omega t + f\sin\nu t.$$

In order to solve this equation, we consider $A \approx \mathrm{const}$ and we seek for a solution of the form

$$x_1 = d_1\sin 3\omega t + d_2\sin\nu t$$

and we obtain

$$x_1 = -\frac{A^3\left(\beta - \gamma\omega^2\right)}{32}\sin 3\omega t + \frac{f}{\omega^2 - \nu^2}\sin\nu t. \tag{3.33}$$

From (3.26), (3.32), (3.33) we deduce the final solution

$$x = A\cos\omega t - \frac{\mu A^3\left(\beta - \gamma\omega^2\right)}{32}\sin 3\omega t + \frac{\mu f}{\omega^2 - \nu^2}\sin\nu t,$$

with A given by (3.32). Remark the term $x_{1o} = A(t)\cos\omega t$, which results from the homogeneous linear equation, the perturbing term $-\left[\mu A^3\left(\beta - \gamma\omega^2\right)/32\right]\sin 3\omega t$ and the term $\left[\mu f/\left(\omega^2 - \nu^2\right)\right]\sin\nu t$ which is due to the excitation. We also remark that if $\nu \leftrightarrow \omega$, then, besides the the self-oscillations, we have resonance and hence, instability. Remark that, by using using the linearization and asymptotic theories, the nonlinear terms $f(x, \dot{x})$ can introduce in the final solution harmonic, sub-harmonic or super-harmonic therms which can make a resonance with the oscillation induced by $\ddot{x} + \omega^2 x$. It is the case of $-\left[\mu A^3\left(\beta - \gamma\omega^2\right)/32\right]\sin 3\omega t$. These forced appearances request separated solutions [2], [3].

We have $\lim_{t\to\infty} A(t) = A^* = \sqrt{4\alpha/\left(\beta + 3\gamma\omega^2\right)}$ with self-oscillations and $|x(t)| \le 2 + \mu/4 + \mu k/\left(\omega^2 - \nu^2\right)$, $|y(t)| \le M$. This means that the elongation is bounded, and that we have a simple stability in the point $O^*(0,0)$ which is a simply stable center.

In order to determine the so called *resonance curve*, we return to equation (3.24) and we apply the variation of constants method. We have

$$x = a\cos\tau + b\sin\tau, \tag{3.34}$$

$$a' \cos \tau + b' \sin \tau = 0, \tag{3.35}$$

$$x' = -a \sin \tau + b \cos \tau, \quad x'' = -a' \sin \tau + b' \cos \tau - a \cos \tau - b \sin \tau. \tag{3.36}$$

By replacing (3.34), (3.36) in (3.24), we obtain a linear algebraic equation in (a', b'). By solving the linear system formed by this equation and equation (3.35) and by averaging the obtained relations with respect to the period $T = 2\pi$, we obtain

$$\begin{cases} a' = \dfrac{\mu}{2} \left[\chi b + \lambda a - \frac{a}{4} \left(a^2 + b^2 \right) (\varepsilon + 3\delta) - f_0 \right], \\ b' = \dfrac{\mu}{2} \left[-\chi a + \lambda b - \frac{b}{4} \left(a^2 + b^2 \right) (\varepsilon + 3\delta) \right]. \end{cases} \tag{3.37}$$

By applying the transformations

$$\tau = \tau_1 \frac{2}{\mu}, \quad X = a \sqrt{\frac{\varepsilon + 3\delta}{4}}, \quad Y = b \sqrt{\frac{\varepsilon + 3\delta}{4}}$$

to system (3.37), we get the system

$$\begin{cases} \dfrac{\partial X}{\partial \tau_1} = \lambda X + \chi Y - X \left(X^2 + Y^2 \right) - F, \\ \dfrac{\partial Y}{\partial \tau_1} = -\chi X + \lambda Y - Y \left(X^2 + Y^2 \right), \end{cases} \tag{3.38}$$

with $F = f_0 \sqrt{(\varepsilon + 3\delta)/4}$. Let us consider the polar coordinates

$$X = A \cos \theta, \quad Y = A \sin \theta.$$

System (3.38) becomes

$$\begin{cases} \dfrac{\partial A}{\partial \tau_1} = \lambda A - A^3 - F \cos \theta \equiv P(A, \theta), \\ \dfrac{\partial \theta}{\partial \tau_1} = -\chi + \frac{F \sin \theta}{A} \equiv Q(A, \theta). \end{cases}$$

Now, we consider the algebraic system

$$\begin{cases} P(A, \theta) = 0, \\ Q(A, \theta) = 0, \end{cases} \tag{3.39}$$

with unknowns (A, θ). By eliminating θ between the equations of system (3.39) we obtain the resonance curve

$$A^2 \left[\chi^2 + \left(\lambda - A^2 \right)^2 \right] = F^2. \tag{3.40}$$

The study of the nonlinear system (3.38) which depends on parameters λ, F and of the resonance curve (3.40) can be numerically done in the plane $\left(U = \chi^2, V = A^2 \right)$. The critical points are solutions of the algebraic system $P(U, V) = 0$, $Q(U, V) = 0$. They can be numerically determined by giving different values to F.

The linearised differential system of (3.38) has the following characteristic equation

$$\begin{vmatrix} \dfrac{\partial P}{\partial A} - \rho & \dfrac{\partial P}{\partial \theta} \\ \\ \dfrac{\partial Q}{\partial A} & \dfrac{\partial Q}{\partial \theta} - \rho \end{vmatrix} = 0 \Leftrightarrow \rho^2 - s\rho + p = 0,$$

where $s = 2\lambda - 4A^2$, $p = 3A^4 - 4\lambda A^2 + \lambda^2 + \chi^2$. The equivalent conditions for asymptotic stability in the point (U, V) are $s < 0$, $p > 0$, i. e.

$$\begin{cases} V > \frac{\lambda}{2}, \\ U > -3V^2 + 4\lambda V - \lambda^2. \end{cases}$$

Consequently, the shaded zone bounded by the parabola $U = -3V^2 + 4\lambda V - \lambda^2$ and the straight lines $V = \frac{\lambda}{2}$, $U = 0$ will be a zone of instability and hence a zone of resonance (see FIGURE 1). We will search the values of F for which the corresponding resonance curve pass trough the instability zone. In the plane (U, V), the resonance curve (3.40) has the following equation

$$V\left[U + (\lambda - V)^2\right] - F^2 = 0,$$

therefore

$$\frac{dV}{dU} = -\frac{V}{3V^2 - 4\lambda V + \lambda^2 + U}.$$

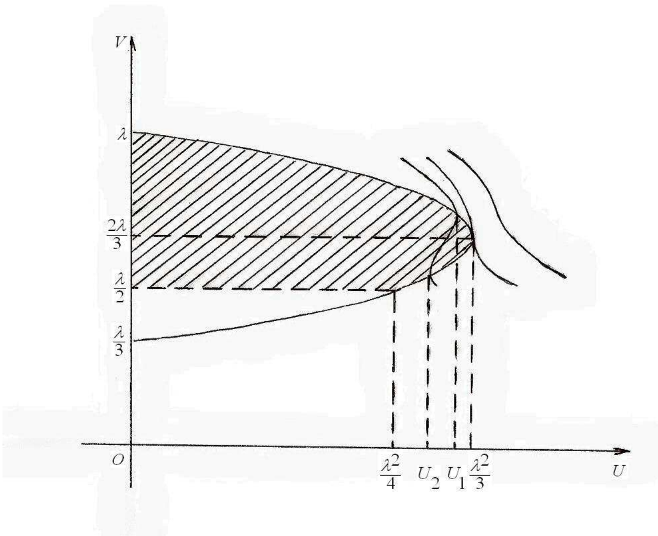

FIGURE 1

Our aim is to determine the points (U, V) of the resonance curve for which $\frac{dV}{dU} = \pm\infty$, i.e. the points of the resonance curve for which the tangents to the resonance curve in these points are vertical. Hence, we have to solve the algebraic system

$$\begin{cases} V\left[U+(\lambda-V)^2\right]-F^2=0, \\ 3V^2-4\lambda V+\lambda^2+U=0. \end{cases} \tag{3.41}$$

The solutions of (3.41) are the intersection points between the resonance curve and the parabola $U=-3V^2+4\lambda V-\lambda^2$. By eliminating U in (3.41), we obtain the following third degree equation

$$-2V^3+2\lambda V^2-F^2=0, \text{with } V\geq 0.$$

It is easy to see that this equation admits real nonnegative roots if and only if

$$F^2\leq \frac{8\lambda^3}{27}.$$

To be more precise, we have

- if $F^2<\frac{8\lambda^3}{27}$, then the resonance curve intersects the parabola $U=-3V^2+4\lambda V-\lambda^2$ in two distinct points,
- if $F^2=\frac{8\lambda^3}{27}$, then the resonance curve intersects the parabola $U=-3V^2+4\lambda V-\lambda^2$ in its vertex,
- if $F^2>\frac{8\lambda^3}{27}$, then the resonance curve does not intersect the parabola $U=-3V^2+4\lambda V-\lambda^2$ (see FIGURE 1).

Remark 3.2. If $U<U_2$ or $U>U_1$, then we have periodic, normal oscillations (stable oscillations). If $U_2<U<U_1$, then we have resonance because the amplitude $V=A^2$ increases in this interval.

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TRANSILVANIA UNIVERSITY OF BRASOV
DEPARTMENT OF ALGEBRA, GEOMETRY AND DIFFERENTIAL EQUATIONS
IULIU MANIU 50
500091 BRASOV, ROMANIA
E-mail address: m.lupu@unitbv.ro
E-mail address: isaiaflorin@yahoo.com