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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

Minimal surfaces

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ABSTRACT. Tensioned membrane constructions have many good physical qualities besides that they are architectonical very interesting. Their construction can start from minimal surfaces. Computation of inner points of surface with minimal area from prescribed boarder is quite difficult in many respects. I was inspired by Spanish research by Juan Monterde who focused on Plateau-Bézier problem in 2002, [1].

Minimal surfaces are constructed approximately during construction in engineering. Therefore we are looking for means that can approximate the searched surface with respect to the best accession to minimal surface area.

One possibility to solve such problem is to investigate surfaces that are parameterized by polynomial functions, in particular by Bézier surfaces, or by piecewise polynomial functions.

1. INTRODUCTION

Tilt constructions prompted me to this work, FIGURE 1. The tilt construction have (besides they are architectural interesting, light and portable) a lot of good physical qualities. Surface film is able to prevent the tilt from permeating of UV radiation or conversely to attain transparency. The tilt constructions came from theory of minimal surfaces. In each point of the minimal surface mean curvature is zero. Thus the curvature of two curves (in two perpendicular planes which are moreover perpendicular to the tangential plane in the done point) are equal but with opposite sign. We can imagine each such point as a saddle point, FIGURE 2.

FIGURE 1. Tilt constructions

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Let the polynomial curve be set or equivalently said control points of boundary of Bézier surface. Let inner control points be found in such way that the resulting Bézier surface has minimal surface area among all Bézier surfaces with the same boundary control points. We are looking for the minimal Bézier surface in the form:

$$\mathbf{x}(u,v)=\sum_{i,j=0}^{n,m} B_i^n(u)B_j^m(v)P_{ij}, \tag{1.1}$$

where $B_i^n(u)=\binom{n}{i}(1-u)^{n-i}u^i$ are Bernstein polynomials and P_{ij} are control net vertices in $\mathbb{R}^3$ of this surface. Control points P_{ij} are set, for $i\in\{0,n\}$, $j\in\{0,m\}$, in parameterization (1.1) for closed spatial boundary curve Γ. Looking for a minimum of area functional is the most natural:

$$A_\Omega(\mathbf{x})=\int_\Omega \|\mathbf{x}_u\times\mathbf{x}_v\| = \int_\Omega \sqrt{\|\mathbf{x}_u\|^2\|\mathbf{x}_v\|^2-(\mathbf{x}_u,\mathbf{x}_v)}dudv=\int_\Omega \sqrt{EG-F^2}dudv, \tag{1.2}$$

where $\mathbf{x}_u$, resp. $\mathbf{x}_v$, are partial derivatives of function $\mathbf{x}$ with respect to variable u, resp. v, and E, F, G are components of the first fundamental form. More precise definition is in [2]. This problem leads to solve of system of nonlinear equations. We can estimate integral (1.2) from the top with aid of simple inequality $\sqrt{EG-F^2}\le\sqrt{EG}\le(E+G)/2$ by Dirichlet integral:

$$A_\Omega(\mathbf{x})\le D_\Omega(\mathbf{x})=\frac{1}{2}\int_\Omega \|\mathbf{x}_u\|^2+\|\mathbf{x}_v\|^2dudv, \tag{1.3}$$

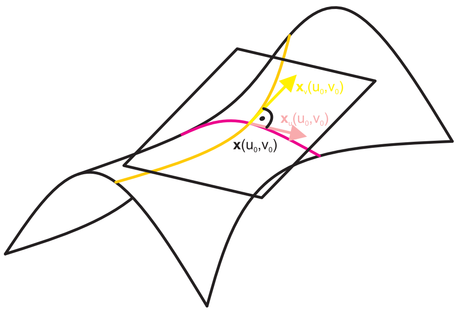

FIGURE 2. Tilt constructions

The function, which minimizes functional (1.3), is harmonic (fits Laplace equation). This results from Dirichlet principal. In the spite of the fact that the set curve is polynomial, it is not assure that the resulting minimal surface with this boundary curve is parameterized by polynomials. Because we are looking for the minimum of functional (1.3) in form (1.1), we are obtaining minimal Bézier surface, which is not minimal for preset boundary curve in Plateau sense. It can be shown that the desired function is not generally harmonic. If the surface is minimal it can be parameterized that the symmetric coordinates arise, for example [3] page 73 or Lichtenstein's theorem in [2], for those the next equations hold true:

$$\|\mathbf{x}_u\| = \|\mathbf{x}_v\| \qquad \text{and} \qquad (\mathbf{x}_u, \mathbf{x}_v) = 0. \tag{1.4}$$

In such case the searching of the minimum of the area functional is the same as the searching of the minimum of Dirichlet's functional. Because points of Bézier surface lie inside of convex hull of the control net vertices, the searching surface is bounded. The searching of the minimum of functional A_Ω is reduced to computing of gradient, which is equal to the zero vector. We get the next by modification of the partial derivative according to the a^{th} coordinate:

$$\frac{\partial D_\Omega}{\partial x_{ij}^a} = \int_\Omega \left(\left(\frac{\partial \mathbf{x}_u}{\partial x_{ij}^a}, \mathbf{x}_u \right) + \left(\frac{\partial \mathbf{x}_v}{\partial x_{ij}^a}, \mathbf{x}_v \right) \right) dudv. \tag{1.5}$$

The partial derivative is obtained by computation:

$$\frac{\partial \mathbf{x}_u}{\partial x_{ij}^a} = n(B_{i-1}^{n-1}(u) - B_i^{n-1}(u))B_j^m(v)e^a. \tag{1.6}$$

where e^a is a^{th} vector of the canonical basis, that is $e^2 = (0,1,0)$. By replacing from (1.6) to (1.5) and by consequential modification, which are precisely worked up for example in [1], we got for $i \in \{1,\ldots,n-1\}$, $j \in \{1,\ldots,m-1\}$ the next equations:

$$\frac{\partial D_\Omega}{\partial x_{ij}^a} = \frac{n^2}{4(n-1)m}\binom{n-1}{i}\binom{m}{j}\sum_{k,l=0}^{n-1,m} C_{ni}^k \frac{\binom{m}{l}}{\binom{2m}{j+l}}(e^a, \triangle^{10} P_{kl}) +$$

$$+\frac{m^2}{4(m-1)n}\binom{n}{i}\binom{m-1}{j}\sum_{k,l=0}^{n,m-1} C_{mj}^l \frac{\binom{n}{k}}{\binom{2n}{i+k}}(e^a, \triangle^{01} P_{kl}), \tag{1.7}$$

where

$$C_{ni}^k = \frac{ni - nk - i}{(n-i)(2n-1-i-k)}\frac{\binom{n-1}{k}}{\binom{2n-2}{i+k-1}},$$

$$\triangle^{10} P_{kl} = P_{k+1,l} - P_{kl} \quad \text{and} \quad \triangle^{01} P_{kl} = P_{k,l+1} - P_{kl}. \tag{1.8}$$

We get $(n-1)(m-1)$ linear equation with the same number of variables. If $m = n$, explanation (1.7) is simpler of course.

The most known polynomial minimal surface is Enneper's surface, FIGURE 3

$$\mathbf{x}(u,v) = (u - \frac{u^3}{3} + uv^2; v - \frac{v^3}{3} + vu^2; u^2 - v^2) \tag{1.9}$$

where the vertices of the control net have in interval $< -1, 1 > \times < -1, 1 >$ coordinates:

$$\begin{array}{cccc} (-\frac{5}{3},-\frac{5}{3},0) & (-1,-\frac{1}{3},-\frac{4}{3}) & (1,-\frac{1}{3},-\frac{4}{3}) & (\frac{5}{3},-\frac{5}{3},0) \\ (-\frac{1}{3},-1,\frac{4}{3}) & (-\frac{5}{9},-\frac{5}{9},0) & (\frac{5}{9},-\frac{5}{9},0) & (\frac{1}{3},-1,\frac{4}{3}) \\ (-\frac{1}{3},1,\frac{4}{3}) & (-\frac{5}{9},\frac{5}{9},0) & (\frac{5}{9},\frac{5}{9},0) & (\frac{1}{3},1,\frac{4}{3}) \\ (-\frac{5}{3},\frac{5}{3},0) & (-1,\frac{1}{3},-\frac{4}{3}) & (1,\frac{1}{3},-\frac{4}{3}) & (\frac{5}{3},\frac{5}{3},0) \end{array}$$

For the closer look at the case of Bézier surface of we can see, FIGURE 4, that the resulting mean curvature are not zero in all points of the surface (the brighter the color, the curvature is closer to zero). From this reason there is more useful to use so called masks. It leads from the biquadratic case, when we are looking only for the one inner point of the surface by solving of equation: $P_{11} = \frac{1}{4}(P_{00}+P_{02}+P_{20}+P_{22})$. It can be rewritten in symbolic form:

$$P_{11} = \frac{1}{4} \times \begin{array}{ccc} 1 & 0 & 1 \\ 0 & \bullet & 0 \\ 1 & 0 & 1 \end{array} . \tag{1.10}$$

By generalization of (1.10) we can choose value α and value β is derived in mask:

$$P_{ij} = \begin{array}{ccc} \alpha & \beta & \alpha \\ \beta & \bullet & \beta \\ \alpha & \beta & \alpha \end{array} . \tag{1.11}$$

With aid of relation $4\alpha + 4\beta = 1$. The computation is simpler and the result is many times better.

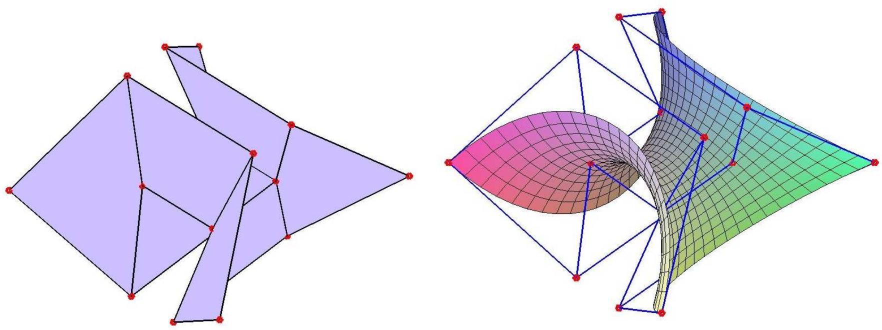

FIGURE 3. Enneper surface. Left: Control net. Right: Final surface.

2. MAIN RESULTS

Let the piecewise polynomial curve be set or equivalently said control points of boundary of the B-spline surface. Let inner control points be found in such way that the resultant B-spline surface has minimal surface area among all B-spline surfaces with the same boundary control points. B-spline surface is defined:

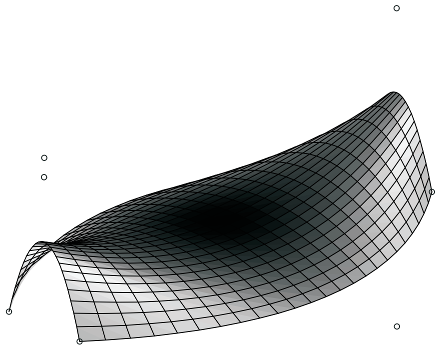

FIGURE 4. Mean curvature of minimal Bézier surface.

$$\mathbf{x}(u,v) = \sum_{i,j=0}^{p,q} N_i^d(u) N_j^e(v) P_{ij}, \tag{2.12}$$

where N_i^d are B-spline basis functions of degree d defined on knot vector $u_0 < u_1 < \ldots < u_m$, respectively N_j^e are B-spline basis functions of degree e defined on knot vector $v_0 < v_1 < \ldots < v_n$, and p, q sign the numbers of control points. Because the resulting surface is polynomial only piecewise and we would like to have first and second derivative in all points continuous (it follows from pure theory about minimal surfaces, when the area integral is computed from second partial derivatives, see [3] or [2]), so let us suppose degrees d, e at least 3. Precise definition you can find in [4], pages 177-178. We copy the same idea as in

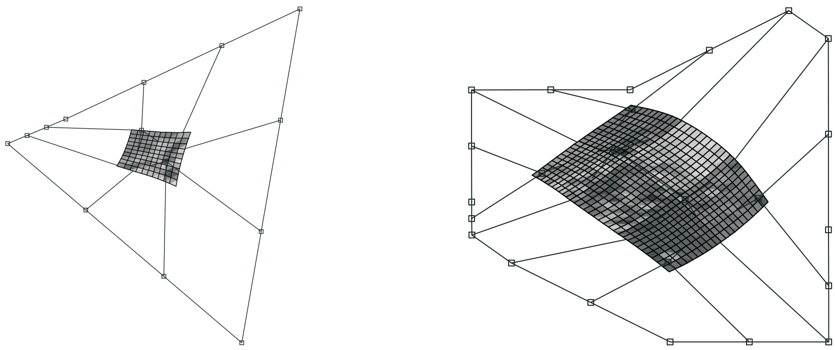

FIGURE 5. Mean curvature of minimal B-spline surface.

the case of Bézier surface. So we found partial derivatives of Dirichlet functional according to the a^{th} coordinate, see (1.5), for B-spline surface. The partial derivative of **x** is obtained by computation (vector of canonical basis will be from this place written in bold to differ from the degree of B-spline basis functions):

$$\frac{\partial \mathbf{x}_u}{\partial x^a_{ij}} = \left(\frac{d}{u_{i+d} - u_i} N_i^{d-1}(u) - \frac{d}{u_{i+1+d} - u_{i+1}} N_{i+1}^{d-1}(u) \right) N_j^e(v) \mathbf{e}^a. \tag{2.13}$$

We got for $i \in \{1, \ldots, n-1\}$, $j \in \{1, \ldots, m-1\}$ the next equations:

$$\begin{aligned}
\frac{\partial D_\Omega}{\partial x^a_{ij}} = \frac{d^2}{u_{i+d} - u_i} & \sum_{\substack{k=1\\l=0}}^{\substack{m\\n}} \frac{(\mathbf{e}^a, P_{kl} - P_{k-1,l})}{u_{k+d} - u_k} \int_R C^{d-1,e}_{ijkl} dudv + \\
- \frac{d^2}{u_{i+1+d} - u_{i+1}} & \sum_{\substack{k=1\\l=0}}^{\substack{m\\n}} \frac{(\mathbf{e}^a, P_{kl} - P_{k-1,l})}{u_{k+d} - u_k} \int_R C^{d-1,e}_{i+1,j,k,l} dudv + \\
+ \frac{e^2}{v_{j+e} - v_j} & \sum_{\substack{k=0\\l=1}}^{\substack{m\\n}} \frac{(\mathbf{e}^a, P_{kl} - P_{k,l-1})}{v_{l+e} - v_l} \int_R C^{d,e-1}_{ijkl} dudv + \\
- \frac{e^2}{v_{j+1+e} - v_{j+1}} & \sum_{\substack{k=0\\l=1}}^{\substack{m\\n}} \frac{(\mathbf{e}^a, P_{kl} - P_{k,l-1})}{v_{l+e} - v_l} \int_R C^{d,e-1}_{i,j-1,k,l} dudv,
\end{aligned} \tag{2.14}$$

where $C^{de}_{ijkl} = N_i^d(u) N_k^d(u) N_j^e(v) N_l^e(v)$ and R is sum of two dimensioned intervals $\langle u_i, u_{i+1} \rangle \times \langle v_j, v_{j+1} \rangle$, $i = 1, \ldots, m-1$, $j = 1, \ldots, n-1$. The surface is piecewise polynomial and we assume that the degrees of the coordinate curves on the surface are greater or equal than 3, thus the derivatives of the basis functions have the degree greater or equal then 2.

FIGURE 5 is colored with respect to the mean curvature. The brighter the color, the smaller the mean curvature. B-spline surface has the color similarly ”grey” in the most points in contrast to Bézier surfaces. This comparison is not very objective, because those surfaces have not got the same boundary. In the first case the purely polynomials on the boundary are discussed and in the second case the boundary is piecewise polynomial.

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