CREATIVE MATH. & INF.
16 (2007), 114 - 123

Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

Unimodal multicriteria optimization via Fibonacci numbers

IOANA CHIOREAN, LIANA LUPŞA and NICOLAE POPOVICI

ABSTRACT. The aim of this work is to develop a numerical method for approximating the efficient sets in multiple criteria optimization problems involving unimodal objective functions. A parallel algorithm corresponding to this method is presented, too.

1. INTRODUCTION

In the papers [7], [6] and [5] the problem of determining or approximating the set of all efficient solutions and the set of all weakly-efficient solutions for a multiple criteria optimization problem, involving generalized unimodal objective function on the feasible set, is treated. These studies are completed with a new method, based on the Fibonacci sequence. The relations between the new method and the methods mentioned above are discussed. Also, a parallel algorithm for approximation the set of all efficient solutions and the set of all weakly-efficient solutions is given.

2. UNIMODAL VECTORIAL FUNCTIONS ON A SET AND SOME OF THEIR PROPERTIES

An extension of the classical concept of unimodality was recently proposed in [4] and it is slightly modified in [5].

Definition 2.1. (see [5]) Let $f : D \to \mathbb{R}$ be a function, defined on a nonempty set $D \subset \mathbb{R}$. We say that f is *lower unimodal on* $S \subset D$ if there exist $u, v \in S$ satisfying the following conditions:

(LU1) $f(u) = f(v)$;
(LU2) $f(x) > f(y)$ whenever $x, y \in S$, $x < y \leq u$;
(LU3) $f(x) < f(y)$ whenever $x, y \in S$, $v \leq x < y$;
(LU4) $S \cap [u, v] = \{u, v\}$.

We remember that if f is lower unimodal on S, then there exists a unique pair $(u, v) \in S \times S$ of numbers satisfying (LU1) – (LU4), implicitly defined by:

$$\operatorname*{Argmin}_{x \in S} f(x) = \{u, v\} \quad \text{and} \quad u \leq v.$$

Received: 10.09.2006. In revised form: 15.11.2006
2000 *Mathematics Subject Classification.* 90C29, 90C10, 65H05, 66Y05.
Key words and phrases. *Generalized unimodal functions, minimum points, multiple criteria programming, efficient points, weakly efficient points, serial and parallel calculus.*

Moreover, when $S = D$ in Definition 2.1 is a compact interval, it follows by (LU4) that $u = v$ and we recover the classical notion of lower unimodality (see, for example [2]). In this case

$$\operatorname*{Argmin}_{x\in S} f(x) = \{u\}.$$

Now, let $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$ ($m \in \mathbb{N}^*$, $m \geq 2$) be a vector-valued function defined on a nonempty set $D \subset \mathbb{R}$.

Definition 2.2. *(see* [5]*)*We said that the function f is *lower unimodal on S* if all its scalar components $f_1, \ldots, f_m$ are lower unimodal on S.

Consider the following multicriteria optimization problem:

$$\begin{cases} \text{Minimize} & f(x) \\ \text{subject to} & x \in S, \end{cases} \tag{2.1}$$

where the partial ordering in the image space of the objective function is understood to be induced by the standard ordering cone $\mathbb{R}^m_+$. More precisely, denoting $I := \{1, \ldots, m\}$, we have for any $a = (a_1, \ldots, a_m), b = (b_1, \ldots, b_m) \in \mathbb{R}^n$:

$$\begin{aligned} a \leq b \quad &:\Longleftrightarrow \quad a_i \leq b_i \quad \text{for all} \quad i \in I \\ a < b \quad &:\Longleftrightarrow \quad a_i < b_i \quad \text{for all} \quad i \in I. \end{aligned}$$

Recall (see e.g. [3]) that the sets of *efficient solutions* and *weakly-efficient solutions* of problem (2.1) are given, respectively, by:

$$\begin{aligned} \operatorname{Eff}(S; f) \;&:=\; \left\{x \in S \mid \left(f(x) - \mathbb{R}^m_+\right) \cap f(S) = \{f(x)\}\right\} \\ &=\; \{x \in S \mid \nexists\, y \in S \text{ such that } f(y) \leqslant f(x) \neq f(y)\}\,, \\ \operatorname{WEff}(S; f) \;&:=\; \left\{x \in S \mid \left(f(x) - \operatorname{int} \mathbb{R}^m_+\right) \cap f(S) = \emptyset\right\} \\ &=\; \{x \in S \mid \nexists\, y \in S \;\text{ such that }\; f(y) < f(x)\}\,. \end{aligned}$$

In the hypothesis that f is lower unimodal on S, in the paper [6], the authors showed that both the sets $\operatorname{Eff}(S; f)$ and $\operatorname{WEff}(S; f)$ can be completely determined only by using the numbers $u_1, v_1, \ldots, u_m, v_m$, where by u_i, v_i we denote, for every $i \in \{1, ..., m\}$, the points u and v from Definition 2.1.

In what follows we suppose that the function $f : [a, b] \to \mathbb{R}^m$, is lower unimodal on $[a, b]$, a and b being real number, $a < b$. Then

$$u_i \;=\; v_i, \text{ for all } i \in \{1, ..., m\}. \tag{2.2}$$

Let us denote:

$$\underline{u} := \min_{i\in I} u_i, \quad \overline{u} := \max_{i\in I} u_i.$$

Remark 2.1. By the theorems 2.1 and 2.2 from [6] it follows that, in the particular case when $S = [a, b]$ is a compact interval and $f : [a, b] \to \mathbb{R}^m$ is lower unimodal on $[a, b]$, we have

$$\operatorname{Eff}([a, b]; f) = \operatorname{WEff}([a, b]; f) = [\underline{u}, \overline{u}].$$

Let now c and d be real numbers, $a \leq c < d \leq b$. We denote by

$$\begin{aligned} I^-_{c,d} \;&=\; \{i \in \{1, ..., m\} \,|\, f_i(c) < f_i(d)\}, \\ I^0_{c,d} \;&=\; \{i \in \{1, ..., m\} \,|\, f_i(c) = f_i(d)\}, \end{aligned}$$

$$I^{+}_{c,d} = \{i \in \{1, ..., m\} \mid f_i(c) > f_i(d)\}.$$

Remark 2.2. From Theorem 2 of [5], we obtain the following results:

(i) If $I^{-}_{c,d} \neq \emptyset$ or $I^{-}_{c,d} = \emptyset$ and $I^{0}_{c,d} \neq \emptyset$, then $\underline{u} \in [a, d]$.

(ii) If $I^{-}_{c,d} = \emptyset$ and $I^{0}_{c,d} = \emptyset$, then $\underline{u} \in [c, b]$.

(iii) If $I^{+}_{c,d} \neq \emptyset$ or $I^{+}_{c,d} = \emptyset$ and $I^{0}_{c,d} \neq \emptyset$, then $\overline{u} \in [c, b]$.

(iv) If $I^{+}_{c,d} = \emptyset$ and $I^{0}_{c,d} = \emptyset$, then $\overline{u} \in [a, d]$.

We will use these results in the next section to elaborate a method for approximate the set of efficient points and the set of all weakly-efficient points.

3. Fibonacci's methods for vectorial function

In what follows we suppose that the function $f : [a, b] \to \mathbb{R}^m$ is lower unimodal on $[a, b]$, where m is a natural number, $m \geq 2$, a and b being real numbers with $a < b$. In this hypotheses we give an algorithm, based on the idea of the Fibonacci's method for approximation the minimum point of a real lower unimodal function. In our algorithm a set EF is built. If $\varepsilon > 0$ is a given error, this set will approximate the sets $\text{Eff}([a, b]; f)$ and $\text{WEff}([a, b]; f)$ (which are equal, in our hypotheses, in view of Remark 2.1) such that we shall have

$$\text{long}(EF \setminus \text{Eff}([a, b]; f)) \leq \varepsilon \tag{3.3}$$

and

$$\text{long}(\text{Eff}([a, b]; f) \setminus EF) \leq \varepsilon. \tag{3.4}$$

If L is a real interval, long(L) denote the length of this interval. If L is a finite union of disjoint intervals, then long(L) denote the sum of the lengths of these intervals.

We mention that an algorithm for approximate the sets $\text{Eff}([a, b]; f)$ and $\text{WEff}([a, b]; f)$, was presented in [7]. It is based on the equidistant cuts technique. Now we give another method, which is more better than the previous one. For this we use the Fibonacci's numbers.

It is known that the Fibonacci numbers F_k, $k \in \mathbb{N}^*$, i.e. the numbers

$$F_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right), \quad \text{for each } k \in \mathbb{N}^*,$$

satisfy the following recurrence formula

$$F_{k+1} = F_k + F_{k-1}, \text{ for each } k \in \mathbb{N}^*, k \geqq 2, \ F_1 = F_2 = 1.$$

VF Algorithm

Step 1. Choose a natural number p, $p \geq 2$, and Let
$k = 1$;
$\underline{a}_1 := a$;
$\overline{a}_1 := a$;
$\underline{b}_1 := b$;
$\overline{b}_1 := b$;

Step 2. Take

$$\underline{c}_k := \underline{a}_k + t_k(\underline{b}_k - \underline{a}_k); \tag{3.5}$$

$$\underline{d}_k := \underline{b}_k - t_k(\underline{b}_k - \underline{a}_k); \tag{3.6}$$

where t_k is given by

$$t_k = \frac{F_{p-k+1}}{F_{p-k+3}}; \tag{3.7}$$

Step 3. If

$$\underline{I}_k \ = \ I^{-}_{\underline{c}_k,\underline{d}_k} \bigcup I^{0}_{\underline{c}_k,\underline{d}_k} \neq \emptyset \tag{3.8}$$

then Let
$\underline{a}_{k+1} := \underline{a}_k$;
$\underline{b}_{k+1} := \underline{d}_k$;
$\underline{u}_k := \underline{c}_k$;
else Let
$\underline{a}_{k+1} := \underline{c}_k$;
$\underline{b}_{k+1} := \underline{b}_k$;
$\underline{u}_k := \underline{d}_k$;

Step 4. Take

$$\overline{c}_k := \overline{a}_k + t_k(\overline{b}_k - \overline{a}_k); \tag{3.9}$$

$$\overline{d}_k := \overline{b}_k - t_k(\overline{b}_k - \overline{a}_k); \tag{3.10}$$

where t_k is given by (3.7).

Step 5. If

$$\overline{I}_k \ = \ I^{+}_{\overline{c}_k,\overline{d}_k} \bigcup I^{0}_{\overline{c}_k,\overline{d}_k} \ \neq \ \emptyset \tag{3.11}$$

then Let
$\overline{a}_{k+1} := \overline{c}_k$;
$\overline{b}_{k+1} := \overline{b}_k$;
$\overline{u}_k := \overline{d}_k$;
else Let
$\overline{a}_{k+1} := \overline{a}_k$;
$\overline{b}_{k+1} := \overline{d}_k$;
$\overline{u}_k := \overline{c}_k$;

Step 6. If $k < p$ then
Increase k by 1 and Go back to step 2;
else Proceed.

Step 7. Let
EF $:= [\underline{u}_p, \overline{u}_p]$;
and Stop.

Remark 3.3. In view of the properties of the Fibonacci numbers, we get that

$$\underline{d}_k = \underline{b}_k - t_k(\underline{b}_k - \underline{a}_k) \ = \ \underline{a}_k + \frac{F_{p-k+2}}{F_{p-k+3}}(\underline{b}_k - \underline{a}_k),$$

and

$$\overline{d}_k = \overline{b}_k - t_k(\overline{b}_k - \overline{a}_k) \ = \ \overline{a}_k + \frac{F_{p-k+2}}{F_{p-k+3}}(\overline{b}_k - \overline{a}_k).$$

We remark that, if $k \in \{1, ..., p-1\}$,

$$0 < \frac{F_{p-k+1}}{F_{p-k+3}} = \frac{F_{p-k+1}}{F_{p-k+2} + F_{p-k+1}} < \frac{F_{p-k+1}}{2F_{p-k+1}} = \frac{1}{2}.$$

Then $0 < t_k < 1/2$, for all $k \in \{1, ..., p-1\}$. Therefore, for all $k \in \{1, ..., p-1\}$, we have

$$\underline{a}_k < \underline{c}_k < \underline{d}_k < \underline{b}_k, \tag{3.12}$$

$$\overline{a}_k < \overline{c}_k < \overline{d}_k < \overline{b}_k. \tag{3.13}$$

If $k = p$, then

$$t_p = \frac{F_{p-p+1}}{F_{p-p+3}} = \frac{1}{2}$$

and, therefore $\underline{c}_p = \underline{d}_p$ and $\overline{c}_p = \overline{d}_p$.

From Remark 3.3 and from steps 2 and 4 we get that, for all $k \in \{2, ..., p\}$, we have

$$\begin{aligned}
\underline{d}_k &= \underline{c}_{k-1}, \quad \text{if } I^-_{\underline{c}_{k-1},\underline{d}_{k-1}} \bigcup I^0_{\underline{c}_{k-1},\underline{d}_{k-1}} \neq \emptyset, \\
\underline{c}_k &= \underline{d}_{k-1}, \quad \text{if } I^-_{\underline{c}_{k-1},\underline{d}_{k-1}} \bigcup I^0_{\underline{c}_{k-1},\underline{d}_{k-1}} = \emptyset, \\
\overline{c}_k &= \overline{d}_{k-1}, \quad \text{if } I^+_{\overline{c}_{k-1},\underline{d}_{k-1}} \neq \emptyset \bigcup I^0_{\underline{c}_{k-1},\underline{d}_{k-1}} \neq \emptyset, \\
\overline{d}_k &= \overline{c}_{k-1}, \quad \text{if } I^+_{\overline{c}_{k-1},\overline{d}_{k-1}} \bigcup I^0_{\overline{c}_{k-1},\overline{d}_{k-1}} = \emptyset.
\end{aligned}$$

These remarks are very important because they underline the fact that *at every iteration, except the first, the numbers of the evaluations of the function f is equal to 1.*

Remark 3.4. For every $k \in \{1, ..., p\}$ we have

$$\underline{b}_{k+1} - \underline{a}_{k+1} = \frac{F_{p-k+2}}{F_{p-k+3}}(\underline{b}_k - \underline{a}_k) = \frac{F_{p-k+2}}{F_{p+2}}(b-a) \tag{3.14}$$

and

$$\overline{b}_{k+1} - \overline{a}_{k+1} = \frac{F_{p-k+2}}{F_{p-k+3}}(\overline{b}_k - \overline{a}_k) = \frac{F_{p-k+2}}{F_{p+2}}(b-a) \tag{3.15}$$

Indeed, if $I^-_{\underline{c}_k,\underline{d}_k} \bigcup I^0_{\underline{c}_k,\underline{d}_k} \neq \emptyset$, then, using Remark 3.3 we get

$$\underline{b}_{k+1} - \underline{a}_{k+1} = \underline{d}_k - \underline{a}_k = \underline{a}_k + \frac{F_{p-k+2}}{F_{p-k+3}}(\underline{b}_k - \underline{a}_k) - \underline{a}_k = \frac{F_{p-k+2}}{F_{p-k+3}}(\underline{b}_k - \underline{a}_k).$$

If $I^-_{\underline{c}_k,\underline{d}_k} \bigcup I^0_{\underline{c}_k,\underline{d}_k} = \emptyset$, then

$$\underline{b}_{k+1} - \underline{a}_{k+1} = \underline{b}_k - \underline{c}_k = \frac{F_{p-k+2}}{F_{p-k+3}}(\underline{b}_k - \underline{a}_k).$$

Therefore

$$\underline{b}_{k+1} - \underline{a}_{k+1} = \frac{F_{p-k+2}}{F_{p-k+3}}(\underline{b}_k - \underline{a}_k). \tag{3.16}$$

It follows that we have

$$\underline{b}_k - \underline{a}_k = \frac{F_{p-k+3}}{F_{p-k+4}}(\underline{b}_{k-1} - \underline{a}_{k-1}).$$

$$\underline{b}_{k-1} - \underline{a}_{k-1} = \frac{F_{p-k+4}}{F_{p-k+5}}(\underline{b}_{k-2} - \underline{a}_{k-2}).$$

$$\ldots$$

$$\underline{b}_4 - \underline{a}_4 = \frac{F_{p-1}}{F_p}(\underline{b}_3 - \underline{a}_3).$$

$$\underline{b}_3 - \underline{a}_3 = \frac{F_p}{F_{p+1}}(\underline{b}_2 - \underline{a}_2).$$

$$\underline{b}_2 - \underline{a}_2 = \frac{F_{p+1}}{F_{p+2}}(\underline{b}_1 - \underline{a}_1) = \frac{F_{p+1}}{F_{p+2}}(b - a).$$

Multiplying, member by member, the above equalities, we obtain

$$\underline{b}_{k+1} - \underline{a}_{k+1} = \frac{F_{p-k+2}}{F_{p+2}}(b - a).$$

Hence, in the both cases we obtain that (3.14) holds.

Analogous we can prove that (3.15) is true.

Theorem 3.1. *If a and b are real numbers, $a < b$, m is a natural number, $m \geq 2$, the function $f : [a, b] \to \mathbb{R}^m$ is lower unimodal on $[a, b]$, $\varepsilon > 0$ is a real number, p is a natural number, $p \geq 2$, and $\underline{a}_1, ..., \underline{a}_p$, $\underline{b}_1, ..., \underline{b}_p$, $\underline{u}_1, ..., \underline{u}_p$, are the points given by the VF Algorithm, then for every $k \in \{1, ..., p\}$ we have:*

$$\underline{u} \in [\underline{a}_k, \underline{b}_k], \tag{3.17}$$

$$|u_k - \underline{u}| \leq \frac{F_{p-k+2}}{F_{p+2}}(b - a). \tag{3.18}$$

If, in addition, the number p is chosen such that

$$\frac{b - a}{F_{p+2}} < \frac{\varepsilon}{2}, \tag{3.19}$$

then

$$|\underline{u} - \underline{u}_p| < \frac{\varepsilon}{2}. \tag{3.20}$$

Proof. i) First we prove that $\underline{u} \in [a_1, b_1]$. Indeed, Step 1 gives $\underline{a}_1 = a$ and $\underline{b}_1 = b$. Because $\underline{u} \in [a, b]$, it follows $\underline{u} \in [\underline{a}_1, \underline{b}_1]$.

Now we prove that if $k \in \{1, ..., p\}$, then $\underline{u} \in [\underline{a}_{k+1}, \underline{b}_{k+1}]$ and $|u_k - \underline{u}| \leq t_{k+1}(b_k - a_k)$.

Let $\underline{c}_k$ and $\underline{d}_k$ be the points chosen at the k^{th} iteration. Two cases are possible:

1. $I^-_{\underline{c}_k, \underline{d}_k} \bigcup I^0_{\underline{c}_k, \underline{d}_k} \neq \emptyset$;
2. $I^-_{\underline{c}_k, \underline{d}_k} \bigcup I^-_{\underline{c}_k, \underline{d}_k} = \emptyset$.

Let consider $I^-_{\underline{c}_k, \underline{d}_k} \bigcup I^0_{\underline{c}_k, \underline{d}_k} \neq \emptyset$. From Step 3 it results

$$\underline{a}_{k+1} = \underline{a}_k, \quad \underline{b}_{k+1} = \underline{d}_k, \quad \underline{u}_k = \underline{c}_k. \tag{3.21}$$

On the other hand, Remark 2.2 implies $\underline{u} \in [\underline{a}_k, \underline{d}_k]$. Therefore $\underline{u} \in [\underline{a}_{k+1}, \underline{b}_{k+1}]$ and $\underline{u}_k \in [\underline{a}_{k+1}, \underline{b}_{k+1}]$. These imply

$$|\underline{u} - \underline{u}_k| \leq \underline{b}_{k+1} - \underline{a}_{k+1}.$$

But, in view of Remark 3.4 we have

$$\underline{b}_{k+1} - \underline{a}_{k+1} = \frac{F_{p-k+2}}{F_{p+2}}(b - a).$$

Hence

$$|\underline{u} - \underline{u}_k| \le \frac{F_{p-k+2}}{F_{p+2}}(b-a).$$

Therefore (3.17) and (3.18) are true. If $I^-_{\underline{c}_k,\underline{d}_k} \bigcup I^0_{\underline{c}_k,\underline{d}_k} = \emptyset$, from Step 3 we get

$$\underline{a}_{k+1} = \underline{c}_k, \quad \underline{b}_{k+1} = \underline{b}_k, \quad \underline{u}_k = \underline{d}_k. \tag{3.22}$$

On the other hand, Remark 2.2 implies $\underline{u} \in [\underline{c}_k, \underline{b}_k]$. Therefore $\underline{u} \in [\underline{a}_{k+1}, \underline{b}_{k+1}]$ and $\underline{u}_k \in [\underline{a}_{k+1}, \underline{b}_{k+1}]$ These imply

$$|\underline{u} - \underline{u}_k| \le \underline{b}_{k+1} - \underline{a}_{k+1} = \frac{F_{p-k+2}}{F_{p+2}}(b-a).$$

Hence (3.17) and (3.18) are also true.

If, in addition, the number p is chosen such that (3.19) holds, we obtain

$$|\underline{u} - \underline{u}_p| < \varepsilon/2.$$

In the same manner we can prove the following result:

Theorem 3.2. *If a and b are real numbers, $a < b$, m is a natural number, $m \ge 2$, the function $f : [a, b] \to \mathbb{R}^m$ is lower unimodal on $[a, b]$, $\varepsilon > 0$ is a real number, p is a natural number, $p \ge 2$, $\overline{a}_1, ..., \overline{a}_p$, $\overline{b}_1, ..., \overline{b}_p$, $\overline{u}_1, ..., \overline{u}_p$, are the points given by the VF Algorithm and $k \in \{1, ..., p\}$, then:*

$$\overline{u} \in [\overline{a}_k, \overline{b}_k], \tag{3.23}$$

$$|\overline{u}_k - \overline{u}| \le t_{k+1}(\overline{b}_k - \overline{a}_k). \tag{3.24}$$

If, in addition, the number p is chosen such that (3.19) holds, then

$$|\overline{u} - \overline{u}_p| < \frac{\varepsilon}{2}. \tag{3.25}$$

Corollary 3.1. *If a and b are real numbers, $a < b$, m is a natural number, $m \ge 2$, the function $f : [a, b] \to \mathbb{R}^m$ is lower unimodal on $[a, b]$, $\varepsilon > 0$ is a real number, p is a natural number, $p \ge 2$, $\underline{a}_1, ..., \underline{a}_p$, $\underline{b}_1, ..., \underline{b}_p$, $\underline{u}_1, ..., \underline{u}_p$, $\overline{a}_1, ..., \overline{a}_p$, $\overline{b}_1, ..., \overline{b}_p$, $\overline{u}_1, ..., \overline{u}_p$ are the points given by the VF Algorithm, then:*

i) *For every $k \in \{1, ..., p\}$,*

$$\text{Eff}([a, b]; f) \subseteq [\underline{a}_k, \overline{b}_k]. \tag{3.26}$$

ii) *If the number p is chosen such that (3.19) holds, then*

$$long(EF \setminus \text{Eff}([a, b]; f)) \le \varepsilon \quad and \quad long(\text{Eff}([a, b]; f) \setminus EF) \le \varepsilon. \tag{3.27}$$

Proof. i) As $\underline{u} \in [\underline{a}_k, \underline{b}_k]$, $\overline{u} \in [\overline{a}_k, \overline{b}_k]$ and $\text{Eff}([a, b]; f) = [\underline{u}, \overline{u}]$, obviously (3.26) is true.

ii) In view of (3.20) and (3.25), we obtain

$$\begin{aligned} \text{long}(EF \setminus \text{Eff}([a, b]; f)) &= \text{long}([\underline{u}_p, \overline{u}_p] \setminus [\underline{u}, \overline{u}]) \\ &\le |\underline{u}_p - \underline{u}| + |\overline{u}_p - \overline{u}| < \varepsilon. \end{aligned}$$

Analogously, we deduce that

$$\begin{aligned} \text{long}(\text{Eff}([a, b]; f) \setminus EF) &= \text{long}([\underline{u}, \overline{u}] \setminus [\underline{u}_p, \overline{u}_p]) \\ &\le |\underline{u}_p - \underline{u}| + |\overline{u}_p - \overline{u}| < \varepsilon. \end{aligned}$$

Example 3.1. Let $f : [0, 5] \to \mathbb{R}^3$,

$$f(x) = (|x-3|,\, x^2 - x,\, |x-2|), \text{ for all } x \in [0, 5]$$

We shall determine the set of all efficient solutions of the multicriterial problem (MUP) using the VF Algorithm with an error $\varepsilon = 0.1$. Obviously, the function f is lower unimodal on $[0, 5]$. The condition (3.19) implies $p = 8$. If we apply the VF Algorithm we obtain (see Table 1) $EF = \left[\frac{5}{11}, 3\right]$.

4. A parallel algorithm

Due to the fact that the computing of the numbers $\underline{u}$ and $\overline{u}$ involve mainly the same type of operations, these computations may be performed simultaneously. So, we consider a routing network in which an element is connected with a selected number of others (see [1]). For instance, in order to perform in parallel the VF algorithm, we may consider such a network, in which two "master" processors (with ID numbers 1 and 2) are connected with, respectively, m processors "slaves". Obviously, the execution will be of "master-slave" type of execution (see [1]). We take into account a Master processor which sends information to some Slaves processors, receives information from them, and then gives the final result.

Remark 4.5. We do not present the whole parallel algorithm, but only emphasize where the simultaneously execution is done.

Let p be a natural number, such that $1/F_{p+2} < \varepsilon$.
The VF parallel algorithm is the following.
"Master" execution:

For $j := 1$ to 2 in parallel do
  $a_{1,j} := a$;
  $b_{1,j} := b$;
$k := 1$;
  Repeat
    Let
      $t_k = \dfrac{F_{p-k+1}}{F_{p-k+3}}$;
      $c_{k,j} := a_{k,j} + t_k(b_{k,j} - a_{k,j})$;
      $d_{k,j} := b_{k,j} - t_k(b_{k,j} - a_{k,j})$;
    Send Message to Slaves $(c_{k,j}, d_{k,j}, j)$;
    Receive Message from Slave $(flag, j)$;
    If $j = 1$
      If $ExistFlag(flag) = 1$ then
        $a_{k+1,j} := a_{k,j}$;
        $b_{k+1,j} := d_{k,j}$;
        $u_{k,j} := c_{k,j}$;
      else
        $a_{k+1,j} := c_{k,j}$;
        $b_{k+1,j} := b_{k,j}$;
        $u_{k,j} := d_{k,j}$;
    else

If $ExistFlag(flag) = 2$ then
$a_{k+1,j} := c_{k,j}$;
$b_{k+1,j} := b_{k,j}$;
$u_{k,j} := d_{k,j}$;
else
$a_{k+1,j} := a_{k,j}$;
$b_{k+1,j} := d_{k,j}$;
$u_{k,j} := d_{k,j}$;
Until $k = p$;
Let $EF := [u_{p,1},\ u_{p,2}]$;
End.

"Slaves" execution;
Receive Message from Master $(c, d; j)$;
If $j = 1$ then
for $h := 1$ to m in parallel do
if $f_h(c) < f_h(d)$ then flag:=1;
if $f_h(c) = f_h(d)$ then flag:=1;
if $f_h(c) > f_h(d)$ then flag:=-1;
else
for $h := 1$ to m in parallel do
if $f_h(c) > f_h(d)$ then flag:=2;
if $f_h(c) = f_h(d)$ then flag:=2;
if $f_h(c) < f_h(d)$ then flag:=-1;
Send Massage to Master (flag,j);

k	$\underline{a}_k$	$\underline{b}_k$	$\underline{c}_k$	$\underline{d}_k$	$\underline{I}_k$	$\underline{u}_k$	$\overline{a}_k$	$\overline{b}_k$	$\overline{c}_k$	$\overline{d}_k$	$\overline{I}_k$	$\overline{u}_k$
1	0	5	$\frac{21}{11}$	$\frac{34}{11}$	$\neq \emptyset$	$\frac{21}{11}$	0	5	$\frac{21}{11}$	$\frac{34}{11}$	$\neq \emptyset$	$\frac{34}{11}$
2	0	$\frac{34}{11}$	$\frac{13}{11}$	$\frac{21}{11}$	$\neq \emptyset$	$\frac{13}{11}$	$\frac{21}{11}$	5	$\frac{34}{11}$	$\frac{42}{11}$	$\emptyset$	$\frac{34}{11}$
3	0	$\frac{21}{11}$	$\frac{8}{11}$	$\frac{13}{11}$	$\neq \emptyset$	$\frac{8}{11}$	$\frac{21}{11}$	$\frac{42}{11}$	$\frac{29}{11}$	$\frac{34}{11}$	$\neq \emptyset$	$\frac{34}{11}$
4	0	$\frac{13}{11}$	$\frac{5}{11}$	$\frac{8}{11}$	$\neq \emptyset$	$\frac{5}{11}$	$\frac{29}{11}$	$\frac{42}{11}$	$\frac{34}{11}$	$\frac{37}{11}$	$\emptyset$	$\frac{34}{11}$
5	0	$\frac{8}{11}$	$\frac{3}{11}$	$\frac{5}{11}$	$\emptyset$	$\frac{5}{11}$	$\frac{29}{11}$	$\frac{37}{11}$	$\frac{32}{11}$	$\frac{34}{11}$	$\neq \emptyset$	$\frac{34}{11}$
6	$\frac{3}{11}$	$\frac{8}{11}$	$\frac{5}{11}$	$\frac{6}{11}$	$\neq \emptyset$	$\frac{5}{11}$	$\frac{32}{11}$	$\frac{37}{11}$	$\frac{34}{11}$	$\frac{35}{11}$	$\emptyset$	$\frac{34}{11}$
7	$\frac{3}{11}$	$\frac{6}{11}$	$\frac{4}{11}$	$\frac{5}{11}$	$\emptyset$	$\frac{5}{11}$	$\frac{32}{11}$	$\frac{35}{11}$	$\frac{33}{11}$	$\frac{34}{11}$	$\emptyset$	$\frac{33}{11}$
8	$\frac{4}{11}$	$\frac{6}{11}$	$\frac{5}{11}$	$\frac{5}{11}$	$\neq \emptyset$	$\frac{5}{11}$	$\frac{32}{11}$	$\frac{34}{11}$	$\frac{33}{11}$	$\frac{33}{11}$	$\neq \emptyset$	$\frac{33}{11}$

Table 1. Results of VF Algorithm

We remark that, if we use the parallel algorithm, obviously, the execution time is much diminished, which was the purpose of this approach.

REFERENCES

[1] Ioana Chiorean, *Parallel calculus* (in Romanian), Ed. Albastra, Cluj-Napoca, 2000

[2] Karmanov V. G., *Programmation mathématique*. Editions Mir, Moscou, 1977

[3] Luc D. T., *Theory of Vector Optimization*. Springer-Verlag, Berlin, 1989

[4] Lupşa L. and Blaga L. R., *Optimum points and integer unimodal functions*. Automation Computers Applied Mathematics, **13** (2004), No. 1, pp. 21-30

[5] Lupşa L. and Popovici N., *A new algorithm for solving multicriteria unimodal optimization problems*. Annals of "Tiberiu Popoviciu" Seminar of Functional Equations, Approximation and Convexity, **3** (2005), pp. 123-130

[6] Lupşa L. and Popovici N., *Generalized Unimodal Multicriteria Optimization*. Rev. Anal. Numér. Théor Approx., **35** (2006), No. 1, pp. 65-70

[7] Popovici N., Multicriteria optimization with unimodal objective functions, in *Approximation and Optimization. Proceedings of the International Conference on Approximation and Optimization - ICAOR. Cluj-Napoca (Romania), July 29-August 1, 1996*, Vol. 1, Litografia Universităţii Babeş-Bolyai, Cluj-Napoca, 1996, pp. 341-344

BABEŞ-BOLYAI UNIVERSITY
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
M. KOGĂLNICEANU 1
400084 CLUJ-NAPOCA, ROMANIA
E-mail address: ioana@math.ubbcluj.ro
E-mail address: llupsa@math.ubbcluj.ro
E-mail address: popovici@math.ubbcluj.ro