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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

On monotonic and translation-invariant multiresolution analysis constructions

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ABSTRACT. As a continuation of a previous work on generalizing the multiresolution analysis from the wavelets and applying it in a lattices context, this paper studies multiresolution analysis on subsets of $\mathbb{R}^n$. Of primary interest are multiresolution analysis by monotonic and translation-invariant functions; the paper studies their characterisation, properties, and possible constructions.

1. INTRODUCTION

Morphological transforms are an important tool in image processing. They are somewhat complementary to the linear transforms (gaussian filter and Fourier or wavelets transforms).

Wavelet transforms earned an important place in image processing due to two important features: they can decompose an image into simpler elements, so that we can construct operators that act independently on each such element, and they can be used for producing a series of simpler, "coarser" images, useful for a top-down analysis of the image.

One can completely represent a function by its levelset decomposition. As shown in [6] and refined in [4], one can apply any monotonic morphological transform (subject to a few technical conditions) independently on each levelset of a function, and then rebuild a function from the resulting sets. This way, the decomposition into levelsets plays the same role to the monotonic morphological transforms as the Fourier decompositions to the linear transforms.

On the other hand, one can construct multiresolution analysis based on morphological operators. Such constructions are studied, for example, in [1], [2], [3], and [5].

A brief recall of the morphological operators and of the multiresolution analysis construction is given in the next section of the paper. Section 3 introduces the characterisation of a morphological operator by a family of subsets of the domain, and establishes some relations between properties of this characteristic set and properties of the original morphological operator. Special attention is payed to morphological operators that can form a multiresolution analysis.

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2. MORPHOLOGICAL OPERATORS AND APPLICATIONS

Let $D \subseteq \mathbb{R}^d$ denote the domain on which the images are defined; usually $D = \mathbb{R}^d$ or $D = [0,1]^d$ or even $D = \mathbb{Z}^d$.

As we will study translation-invariant operators, we will require in the following that D be itself translation-invariant, which means that $\forall x, y \in D$, $x - y \in D$.

The transforms we analyse take a subset of the domain D and yield another subset of D; thus, they are generally of the form

$$T : \mathcal{K} \to \mathcal{K}, \tag{2.1}$$

where $\mathcal{K} \subseteq \mathcal{P}(D)$ is a conveniently-chosen subset of the set of all subsets of D. Useful choices for $\mathcal{K}$ are the whole $\mathcal{P}(D)$ and the set of Lesbegue measurable subsets of D.

Again, we will require in the following that $\forall X \in \mathcal{K}$ and $\forall x \in D$ we have $(X + x) \in \mathcal{K}$ and $(-X) \in \mathcal{K}$

The definitions of a morphological transform vary from one source to another, all of them requiring that the transform be of the form in (2.1).

We will call a morphological transform $T : \mathcal{K} \to \mathcal{K}$

- *monotonic*, if $\forall X \subseteq Y$, $T(X) \subseteq T(Y)$;
- *translation-invariant*, if $\forall X \in \mathcal{K}$, $\forall x \in D$, $T(X + x) = T(X) + x$;
- *mirror-invariant*, if $T(-X) = -T(X)$;

Recall from [5] the definition of a multiresolution analysis.

Let I be a set of indices, $I = \mathbb{Z}$ or $I = \mathbb{R}$.

Definition 2.1. $T_s : \mathcal{K} \to \mathcal{K}$, $s \in I$, define a *multiresolution analysis over* $\mathcal{K}$ if:

- if $s \le r$ then $T_s \circ T_r = T_s$;
- with notation $\mathcal{V}_s = T_s(\mathcal{K})$, if $s \le r$ then $\mathcal{V}_s \subseteq \mathcal{V}_r$.

3. THE CHARACTERISTIC SET OF T

Let $T : \mathcal{K} \to \mathcal{K}$ be a monotonic and translation-invariant morphological transform.

Definition 3.2. *We call* the characteristic set of T *the set:*

$$\mathcal{B}_T = \{X \in \mathcal{K} : 0 \in T(X)\} \tag{3.2}$$

The following theorems give the properties of the characteristic set and state that the characteristic set completely defines the morphological transform.

Theorem 3.1. *If $X \in \mathcal{B}_T$ and $Y \supseteq X$, then $Y \in \mathcal{B}_T$, for any $X, Y \in \mathcal{K}$.*

The proof is immediate.

Theorem 3.2. *Suppose $\mathcal{B} \subseteq \mathcal{K}$ verifies that if $X \in \mathcal{B}$ and $Y \supseteq X$ then $Y \in \mathcal{B}$. Then $T : \mathcal{K} \to \mathcal{K}$ given by*

$$T(X) = \{x \in D : (X - x) \in \mathcal{B}\} \tag{3.3}$$

is monotonic and translation-invariant and $\mathcal{B}_T = \mathcal{B}$.

Proof. (1) *Monotonicity:* Let $X, Y \in \mathcal{K}$, $X \subseteq Y$. Then for any $x \in D$, $(X - x) \subseteq (Y - x)$. On the other hand, for any $x \in T(X)$, by (3.3) we have $(X - x) \in \mathcal{B}$. It results that, for any $x \in T(X)$, $(Y - x) \in \mathcal{B}$ and therefore $x \in T(Y)$. Therefore $T(X) \subseteq T(Y)$.

(2) *Translation invariance:* Let $z \in D$. We have:

$$\begin{aligned} T(X+z) &= \{x \in D : ((X+z) - x) \in \mathcal{B}\} = \\ &= \{x \in D : (X - (x - z)) \in \mathcal{B}\} = \\ &= \{y + z \in D : (X - y) \in \mathcal{B}\} = \\ &= T(X) + z \end{aligned}$$

(3) $\mathcal{B}_T = \mathcal{B}$: $X \in \mathcal{B}_T$ is equivalent, because of (3.2), to $0 \in T(X)$ and, because of (3.3), is further equivalent to $(X - 0) \in \mathcal{B}$, or, in other words, $X \in \mathcal{B}$. □

Theorem 3.3. *If $T : \mathcal{K} \to \mathcal{K}$ is monotonic and translation-invariant, then*

$$T(X) = \{x \in D : (X - x) \in \mathcal{B}_T\}$$

Proof. $x \in T(X)$ if and only if $0 = x - x \in T(X - x)$ which by (3.2) is equivalent to $(X - x) \in \mathcal{B}_T$, in other words, $x \in \{x \in D : (X - x) \in \mathcal{B}_T\}$. □

Theorems 3.2 and 3.3 state that there is a one-to-one mapping between the monotonic and translation-invariant operators T and the sets $\mathcal{B}$ having the property that if they contain a set they contain any superset of it.

In view of Theorem 3.1, we can define a characteristic set by a subset, such that the characteristic set can be retrieved by adding supersets of all the elements in the subset. This leads to:

Definition 3.3. We say that the set $\mathcal{B}_0$ spawns the set $\mathcal{B}$ if

$$\mathcal{B} = \{X \in \mathcal{K} : \exists Y \in \mathcal{B}_0 \text{ such that } Y \subseteq X\}$$

The following question arises naturally: Given an operator T, is there a minimal set $\mathcal{B}_0$ spawning $\mathcal{B}_T$? Unfortunately, the answer is no (in the general case), as shown by the following example:

Example 3.1. Let $T(X)$ be the interior of X, in the usual topological sense. T is monotonic and translation-invariant. However, there is no minimal set spawning $\mathcal{B}_T$.

Indeed, $\mathcal{B}_T$ is the set of neighborhoods of 0. A subset $\mathcal{B}_0$ spawning $\mathcal{B}_T$ would be the set of discs centered in 0. However, there is no minimal set $\mathcal{B}_0$ spawning $\mathcal{B}_T$.

Here is a list of classical monotonic and translation-invariant transforms and their characteristic set:

- **erosion:** of radius r: $\mathcal{B}_T$ is spawned by the set containing the disc of center 0 and radius r;
- **dilatation:** of radius r: $\mathcal{B}_T$ is spawned by the set containing as singletons all the points of the disc centered in 0 and of radius r;
- **morphological opening:** of radius r: $\mathcal{B}_T$ is spawned by the set of discs of radius r containing the origin;

morphological closure: of radius r: $\mathcal{B}_T$ is the set of all sets in $\mathcal{K}$ having non-void intersections with all discs of radius r containing the origin;

median filter: of radius r: $\mathcal{B}_T$ is spawned by the set of subsets of the disc centered in origin and of radius r, having the area of at least $\frac{1}{2}\pi r^2$;

topological closure: : $\mathcal{B}_T$ is spawned by the set of sets of the elements of sequences convergent towards 0.

3.1. **Compound operators and characteristic sets.** The following gives the relations between the characteristic sets of monotonic and translation-invariant operators where function composition is involved.

Given $\mathcal{B}_T$ and $\mathcal{B}_U$, where T and U are monotonic translation-invariant operators, $\mathcal{B}_{T\circ U}$ is given by:

$$\mathcal{B}_{T\circ U} = \big\{X \in \mathcal{K} : \{x \in D : (X - x) \in \mathcal{B}_U\} \in \mathcal{B}_T\big\}. \tag{3.4}$$

That is, $\mathcal{B}_T$ is the set of sets X with the property that the positions where they can be translated to yield a set in $\mathcal{B}_T$ form the mirroring of a set in $\mathcal{B}_U$.

Using the above relation, we can state the condition for a monotonic and translation-invariant operator to be idempotent.

T is idempotent if and only if

$$\mathcal{B}_T = \big\{X \in \mathcal{K} : \{x \in D : (X - x) \in \mathcal{B}_T\} \in \mathcal{B}_T\big\},$$

or, equivalently, if

$$X \in \mathcal{B}_T \Leftrightarrow T(X) \in \mathcal{B}_T.$$

Finally, the multiresolution condition of Definition 2.1 can be written as follows: $T_s = T_s \circ T_r$ if and only if

$$\mathcal{B}_{T_s} = \big\{X \in \mathcal{K} : \{x \in D : (X - x) \in \mathcal{B}_{T_r}\} \in \mathcal{B}_{T_s}\big\}$$

or, equivalently, if

$$X \in \mathcal{B}_{T_s} \Leftrightarrow T_r(X) \in \mathcal{B}_{T_s}$$

4. Conclusions

The paper studies the morphological operators from the perspective of constructing multiresolution analysis. The monotonic translation-invariant morphological operators are shown to be fully characterized the *characteristic set* introduced in Section 3. The characteristic set of the classical morphological operators is given. Finally, the effects of operator composition and multiresolution conditions are given in terms of characteristic set operations and properties.

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