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Two dimensional divided differences revisited

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ABSTRACT. The notion of two dimensional divided difference was introduced by Academician T. Popoviciu, in 1934, for the case when the number of abscissas is equal with the number of coordinates. In his famous monograph, D. V. Ionescu recovered the Popoviciu's definition and proved an integral representation for the two dimensional divided difference of n-th order.

In a recent monograph, M. Ivan introduced the notion of two dimensional (m, n)-th order divided difference.

The focus of the present paper is to establish some properties of the two dimensional divided difference of (m, n)-th order and to give a representation of the bivariate Lagrange interpolation polynomial in terms of above divided differences.

1. INTRODUCTION

Let m be a non-negative integer and $[a, b]$ be an interval. Consider the space $C[a, b]$ of all continuous real valued functions defined on $[a, b]$ and the distinct knots $x_j \in [a, b]$ $(i = \overline{0, m})$.

The m-th order divided difference of f on the knots $x_0, x_1, \ldots, x_m$ is defined by:

$$[x_0, \ldots, x_m, f] = \sum_{i=0}^{m} \frac{f(x_i)}{(x_i - x_0)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_m)} \tag{1.1}$$

It is also well known the following recurrence formula:

$$[x_0, \ldots, x_m; f] = \frac{[x_1, \ldots, x_m; f] - [x_0, \ldots, x_{m-1}; f]}{x_m - x_0} \tag{1.2}$$

with $[x_0; f] = f(x_0)$. More at this, from (1.2) follows (1.1) by induction with respect m.

If $f, g \in C[a, b]$, the following Leibniz formula for divided differences

$$[x_0, \ldots, x_m; fg] = \sum_{i=0}^{m} [x_0, \ldots, x_i; f][x_i, \ldots, x_m; g] \tag{1.3}$$

holds.

If $f \in C^m[a, b]$ and there exists $f^{(m+1)}$ on $]a, b[$, it is well known the following "mean value" theorem for divided differences:

$$[x_0, x_1, \ldots, x_m; f] = \frac{f^{(m+1)(\xi)}}{(m+1)!} u(x), \quad \text{where } \xi \in]a, b[\tag{1.4}$$

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Denoting by $L_m : C[a,b] \to C[a,b]$ the Lagrange operator, it is well known the following Lagrange interpolation formula:

$$f(x) = L_m(x, x_0, \ldots, x_m; f) + [x, x_0, \ldots, x_m; f]u(x) \tag{1.5}$$

where

$$L_m(x, x_0, \ldots, x_m; f) = f(x_0) + \sum_{i=1}^{m} [x_0, \ldots, x_i; f](x - x_0) \ldots (x - x_{i-1}) \tag{1.6}$$

is the Lagrange interpolation polynomial of degree m and $u(x) = \prod_{i=0}^{m} (x - x_i)$.
All the above results can be found for example in [2], [7], [8], [9], [10].

The focus of the paper is to extend the mentioned results to the case of two dimensional (bivariate) divided differences.

2. The definition of two dimensional divided differences and immediately properties

Let m, n be given non-negative integers and $D = [a,b] \times [c,d]$ be a bidimensional interval. Consider the set $C(D)$ of all real valued functions, continuous on D.

Let $x_0, \ldots, x_m \in [a,b]$ and $y_0, \ldots, y_n \in [c,d]$ be distinct points.
For any $y \in [c,d]$, we denote by $[x_0, \ldots, x_m; f(x,y)]_x$ the parametric extension of m-th order divided difference, i.e. the m-th order divided difference of the function $f(\cdot, y) : [a,b] \to \mathbb{R}$ with respect to knots $x_0, \ldots, x_m$:

$$[x_0, \ldots, x_m; f(x,y)]_x = \sum_{i=0}^{m} \frac{f(x_i, y)}{(x_i - x_0) \ldots (x_i - x_{i-1})(x_i - x_{i+1}) \ldots (x_i - x_m)}. \tag{2.7}$$

In a similar way, the parametric extension of n-th order divided difference of f is the n-th order divided difference of the function $f(x, *) : [c,d] \to \mathbb{R}$ with respect the knots $y_0, \ldots, y_n$:

$$[y_0, \ldots, y_n; f(x,y)]_y = \sum_{j=0}^{n} \frac{f(x, y_j)}{(y_j - y_0) \ldots (y_j - y_{j-1})(y_j - y_{j+1}) \ldots (y_j - y_m)}. \tag{2.8}$$

Proposition 2.1. *The following equalities*

$$\begin{aligned} &[x_0, \ldots, x_m; [y_0, \ldots, y_n; f(x,y)]_y]_x \\ &= [y_0, \ldots, y_n; [x_0, \ldots, x_m; f(x,y)]_x]_y = \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{f(x_i, y_i)}{u'(x_i) v'(y_j)} \end{aligned} \tag{2.9}$$

hold, where $u(x) = \prod_{i=0}^{m} (x - x_i)$, $v(y) = \prod_{j=0}^{n} (y - y_j)$.

Proof. From (2.8) it is obvious that:

$$\big[x_0,\dots,x_m;[y_0,\dots,y_n;f(x,y)]_y\big]_x=\left[x_0,\dots,x_m;\sum_{j=0}^{n}\frac{f(x,y_j)}{v'(y_j)}\right]_x$$
$$=\sum_{j=0}^{n}\frac{1}{v'(y_j)}\,[x_0,\dots,x_m;f(x,y_j)]=\sum_{j=0}^{n}\frac{1}{v'(y_j)}\sum_{i=0}^{m}\frac{f(x_i,y_j)}{u'(x_i)}$$
$$=\sum_{j=0}^{n}\sum_{i=0}^{m}\frac{f(x_i,y_j)}{u'(x_i)v'(y_j)}=\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{f(x_i,y_j)}{u'(x_i)v'(y_j)}\,.$$

□

Definition 2.1. [6] The (m,n)-th order divided difference of the function $f\in C(D)$ with respect the distinct knots $(x_i,y_j)\in D$ $(i=\overline{0,m},j=\overline{0,n})$ is defined by the equality:

$$\left[\begin{matrix}x_0,\dots,x_m\\ y_0,\dots,y_n\end{matrix};f\right]=\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{f(x_i,y_j)}{u'(x_i)v'(y_j)}\,. \tag{2.10}$$

Proposition 2.2. *The (m,n)-th order divided difference (2.10) verifies the following recurrence relation:*

$$\left[\begin{matrix}x_0,\dots,x_m\\ y_0,\dots,y_n\end{matrix};f\right]=\frac{1}{(x_m-x_0)(y_n-y_0)}\left\{\left[\begin{matrix}x_1,\dots,x_m\\ y_1,\dots,y_n\end{matrix};f\right]\right. \tag{2.11}$$
$$\left.-\left[\begin{matrix}x_1,\dots,x_m\\ y_0,\dots,y_{n-1}\end{matrix};f\right]-\left[\begin{matrix}x_0,\dots,x_{m-1}\\ y_1,\dots,y_n\end{matrix};f\right]+\left[\begin{matrix}x_0,\dots,x_{m-1}\\ y_0,\dots,y_{n-1}\end{matrix};f\right]\right\}.$$

Proof. Using the method of parametric extensions [1], [3], [4], like in the proof of Proposition 2.1 we get successively:

$$\left[\begin{matrix}x_0,\dots,x_m\\ y_0,\dots,y_n\end{matrix};f\right]=[y_0,\dots,y_n;[x_0,\dots,x_m;f(x,y)]_x]_y \tag{2.12}$$
$$=\left[y_0,y_1,\dots,y_n;\frac{[x_1,\dots,x_m;f(x,y)]_x-[x_0,\dots,x_{m-1};f(x,y)]_x}{x_m-x_0}\right]_y$$
$$=\frac{1}{x_m-x_0}\,[y_0,\dots,y_n;[x_1,\dots,x_m;f(x,y)]_x]_y$$
$$=\frac{1}{x_m-x_0}\,[y_0,\dots,y_n;[x_0,\dots,x_{m-1};f(x,y)]_x]_y$$
$$=\frac{1}{x_m-x_0}\,\frac{[y_1,\dots,y_n;[x_1,\dots,x_m;f]_x]_y-[y_0,\dots,y_{n-1};[x_1,\dots,x_m;f]_x]_y}{y_n-y_0}$$
$$=\frac{1}{x_m-x_0}\,\frac{[y_1,\dots,y_n;[x_0,\dots,x_{m-1};f]_x]_y-[y_0,\dots,y_{n-1};[x_0,\dots,x_{m-1};f]_x]_y}{y_n-y_0}$$

$$= \frac{1}{(x_m - x_0)(y_n - y_0)} \left\{ \begin{bmatrix} x_1, \dots, x_m & ; f \\ y_1, \dots, y_n & \end{bmatrix} - \begin{bmatrix} x_1, \dots, x_m & ; f \\ y_0, \dots, y_{n-1} & \end{bmatrix} \right.$$

$$\left. - \begin{bmatrix} x_0, \dots, x_{m-1} & ; f \\ y_1, \dots, y_n & \end{bmatrix} + \begin{bmatrix} x_0, \dots, x_{m-1} & ; f \\ y_0, \dots, y_{n-1} & \end{bmatrix} \right\}.$$

□

Proposition 2.3. *Let $D^{(i,j)}$ denoting the (i,j)-th order partial differentiation operator. If:*
(i) $f \in C^{m,n}([a,b] \times [c,d])$; (ii) *There exists $f^{(m+1,n+1)}$ on $]a,b[\times]c,d[$ then, there exists a point $(\xi,\eta) \in]a,b[\times]c,d[$ such that:*

$$\begin{bmatrix} x_0, \dots, x_m & ; f \\ y_0, \dots, y_n & \end{bmatrix} = \frac{u(x)v(y)}{(m+1)!(n+1)!} \, D^{(m+1,n+1)} f(\xi,\eta). \tag{2.13}$$

Proof. Applying the method of parametric extensions and the mean-value theorem for one dimensional divided differences, we get:

$$\begin{bmatrix} x_0, \dots, x_m & ; f \\ y_0, \dots, y_n & \end{bmatrix} = [y_0, \dots, y_n; [x_0, \dots, x_m; f]_x]_y = \left[y_0, \dots, y_n; \tfrac{D^{(m+1,0)} f(\xi,y)}{(m+1)!} \right]$$
$$= \tfrac{u(x)}{(m+1)!} \left[y_0, \dots, y_n; D^{(m+1,0)} f(\xi,y) \right] = \tfrac{u(x)v(y)}{(m+1)!(n+1)!} \, D^{(m+1,n+1)} f(\xi,\eta).$$ □

3. The Lagrange interpolation polynomial for functions of two variables

The Lagrange interpolation problem in the bivariate case is to find the bivariate polynomial of minim degree which interpolates the function $f \in C(D)$ at the distinct points $(x_i, y_j) \in D$, $i = \overline{0,m}$, $j = \overline{0,n}$.

It is known [1], [2], [3], [4] that the above problem has a unique solution, the bivariate Lagrange polynomial $L_{m,n}f$ associated to the function f and to the knots (x_i, y_j), which has the following representation:

$$L_{m,n} \left(\begin{matrix} x, x_0, \dots, x_m \\ y, y_0, \dots, y_n \end{matrix} ; f \right)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} l_i(x) l_j(y) f(x_i, y_j) \tag{3.14}$$

where

$$l_i(x) = \frac{u(x)}{(x - x_i)u'(x_i)}, \quad u(x) = \prod_{i=0}^{m} (x - x_i) \tag{3.15}$$

$$l_j(y) = \frac{v(y)}{(y - y_j)v'(y_j)}, \quad v(x) = \prod_{j=0}^{n} (y - y_j) \tag{3.16}$$

are the fundamental polynomials of the Lagrange interpolation.

It is known that

$$L_{m,n} = L_m^x L_n^y \tag{3.17}$$

L_m^x, L_n^y being the parametric extensions of the univariate interpolation projector [1], [3].

In what follows we shall present the expression of the polynomial (3.14) using the bivariate divided differences. First, we need some auxiliary results.

Lemma 3.1. *The bivariate divided difference* (2.10) *is the coefficient of $x^m y^n$ of the bivariate Lagrange interpolation polynomial* (3.14).

Proof. From the interpolation conditions $L_{m,n}\left(\begin{matrix} x_i, x_0, ..., x_m \\ y_j, y_0, ..., y_n \end{matrix}\ f\right) = f(x_i, y_j)$, $i = \overline{0,m}$, $j = \overline{0,n}$, it follows:

$$L_{m,n}\left(\begin{matrix} x, x_0, \ldots, x_m \\ y, y_0, \ldots, y_n \end{matrix}\ f\right) = \sum_{i=0}^{m}\sum_{j=0}^{n} A_{ij} x^i y^j.$$

For any $y \in [c,d]$, the coefficient of x^m in the Lagrange parametric extension L_m^x is $[x_0, \ldots, x_m; f(\cdot\,, y)]_x$ while, for any $x \in [a,b]$, the coefficient of y^n in the Lagrange parametric extension L_n^y is $[y_0, \ldots, y_n; f(x, *)]_y$. Taking (3.17) into account, we get

$$A_{m,n} = [y_0, \ldots, y_n; [x_0, \ldots, x_m; f]_x]_y = \left[\begin{matrix} x_0, \ldots, x_m \\ y_0, \ldots, y_n \end{matrix}\ ; f\right].$$

□

Corollary 3.1. *If* $f(x,y) = g(x)h(y)$, $(\forall)(x,y) \in [a,b] \times [c,d]$ *where* $g : [a,b] \to \mathbb{R}$, $h : [c,d] \to \mathbb{R}$, *the following equality*

$$\left[\begin{matrix} x_0, \ldots, x_m \\ y_0, \ldots, y_n \end{matrix}\ ; f\right] = [x_0, \ldots, x_m; g][y_0, \ldots, y_n; h] \tag{3.18}$$

holds.

Proof. Taking (3.17) and the hypotheses into account, we have

$$L_{m,n}\left(\begin{matrix} x, x_0, \ldots, x_m \\ y, y_0, \ldots, y_n \end{matrix}\ ; f\right) = L_m(x, x_0, \ldots, x_m; g) L_n(y, y_0, \ldots, y_n; g).$$

The coefficient of $x^m y^n$ in the left side is $\left[\begin{matrix} x_0, \ldots, x_m \\ y_0, \ldots, y_n \end{matrix}\ ; f\right]$ while in the right side it is $[x_0, \ldots, x_m; g][y_0, \ldots, y_n; h]$. □

Corollary 3.2. *For any non-negative integers* m, n, *any* $(x,y) \in D$ *and any* $(x_i, y_j) \in D$ $(i = \overline{0,m}, j = \overline{0,n})$, *the following equality*

$$\left[\begin{matrix} x_0, \ldots, x_m \\ y_0, \ldots, y_n \end{matrix}\ ; x^m y^n\right] = 1 \tag{3.19}$$

holds.

Proof. One applies Corollary 3.1 to the function $f : D \to \mathbb{R}$ $f(x,y) = x^m y^n$ with $g(x) = x^m$ and $h(y) = y^n$. □

Corollary 3.3. *For any permutation* $\{i_0, \ldots, i_m\} \times \{j_0, \ldots, j_n\}$ *of the set* $\{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$, *the following equality*

$$\left[\begin{matrix} x_{i_0}, \ldots, x_{i_m} \\ y_{i_0}, \ldots, y_{j_n} \end{matrix}\ ; f\right] = \left[\begin{matrix} x_0, \ldots, x_m \\ y_0, \ldots, y_n \end{matrix}\ ; f\right] \tag{3.20}$$

holds.

Proof. Let $f : D \to \mathbb{R}$ be given. It is obvious that

$$L_{mn}\left(\begin{matrix} x, x_{i_0}, \ldots, x_{i_m} \\ y, y_{i_0}, \ldots, y_{i_m} \end{matrix}\ ; f\right) = L_{mn}\left(\begin{matrix} x, x_0, \ldots, x_m \\ y, y_0, \ldots, y_m \end{matrix}\ ; g\right), \quad (\forall)\,(x,y) \in D.$$

Then, we identify the coefficients of $x^m y^n$ from the left and respectively right side. $\square$

Proposition 3.1. *The bivariate Lagrange interpolation polynomial can be represented under the form*

$$L_{mn}\left(\begin{array}{l} x, x_0, \ldots, x_m \\ y, y_0, \ldots, y_m \end{array}; f\right) = f(x_0, y_0) \tag{3.21}$$
$$+ \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{l} x_0, \ldots, x_i \\ y_0, \ldots, y_j \end{array}; f\right](x-x_0)\ldots(x-x_{i-1})(y-y_0)\ldots(y-y_{j-1}).$$

Proof. For any $y \in [c, d]$, taking (1.6) into account, we have:

$$L_m^x(x, x_0, \ldots, x_m; f(x, y)) = f(x_0, y)$$
$$+ \sum_{i=j}^{m}[x_0, \ldots, x_i; f(x, y)]_x(x - x_0)\ldots(x - x_{i-1}).$$

next, we get

$$L_{m,n}\left(\begin{array}{l} x, x_0, \ldots, x_m \\ y, y_0, \ldots, y_n \end{array}; f\right) = L_n^y\left[L_m^x(x, x_0, \ldots, x_m; f(x, y))\right]$$
$$= L_n^y\left(f(x_0, y) + \sum_{i=1}^{m}[x_0, \ldots, x_i(f(x, y)]_x(x - x_0)\ldots(x - x_{i-1})\right) = f(x_0, y_0)$$
$$+ \sum_{j=1}^{n}\sum_{i=1}^{m}[y_0, ..., y_j; [x_0, ..., x_i; f(x, y)]]_y\,(x-x_0)...(x-x_{i-1})(y-y_0)...(y-y_{j-1})$$
$$= f(x_0, y_0) + \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{l} x_0, \ldots, x_i \\ y_0, \ldots, y_j \end{array}; f\right](x-x_0)...(x-x_{i-1})(y-y_0)...(y-y_{j-1}).$$

$\square$

Proposition 3.2. *Let*

$$f(x, y) = L_{m,n}\left(\begin{array}{l} x, x_0, \ldots, x_m \\ y, y_0, \ldots, y_n \end{array}; f\right) + R_{m,n}\left(\begin{array}{l} x \\ y \end{array}; f\right) \tag{3.22}$$

be the Lagrange bivariate interpolation formula. The remainder term can be represented under the form:

$$R_{m,n}\left(\begin{array}{l} x \\ y \end{array}; f\right) = [x, x_0, \ldots, x_m; f(\cdot\,, y)]u(x) \tag{3.23}$$
$$+ [y, y_0, ..., y_n; f(x, *)]v(y) - \left[\begin{array}{l} x, x_0, \ldots, x_m \\ y, y_0, \ldots, y_n \end{array}; f\right]u(x)v(y)$$

for any $(x, y) \in D$.

Proof. According to W. J. Gordon [4], the remainder projector associated to the tensorial product $L_m^x L_n^y$ is the boolean sum $R_m^x \oplus R_n^y$ of the remainder operators R_m^x, R_n^y associated to L_m^x and L_n^y.

But:

$$R_m^x f(x,y) = [x, x_0, \ldots, x_m; f(\cdot\,,y)]u(x)$$
$$R_n^y f(x,y) = [y, y_0, \ldots, y_n; f(x,*)]v(y)$$
$$R_m^x R_n^y f(x,y) = \begin{bmatrix} x, x_0, \ldots, x_m & ; f \\ y, y_0, \ldots, y_n & \end{bmatrix} u(x)v(y)$$

and $R_m^x \oplus R_n^y = R_m^x + R_n^y - R_m^x R_n^y$.
Taking the above relations into account, it follows (**??**). □

Corollary 3.4. *If:*
(i) $f \in C^{(m,n)}([a,b] \times [c,d])$; (ii) *there exists* $D^{(m+1,n+1)} f$ *on* $]a,b[\times]c,d[$
then there exists the points $\xi_1 \in]a,b[$, $\eta_1 \in]c,d[$, $(\xi_2,\eta_2) \in]a,b[\times]c,d[$ *such that:*

$$R_{m,n}\begin{pmatrix} x & ; f \\ y & \end{pmatrix} = \frac{u(x)}{(m+1)!} D^{(m+1,0)} f(\xi_1, y) + \frac{v(y)}{(n+1)!} D^{(0,n+1)} f(x, \eta_1)$$
$$- \frac{u(x)v(y)}{(m+1)!(n+1)!} D^{(m+1,n+1)} f(\xi_2, \eta_2)$$

for any $(x,y) \in [a,b] \times [c,d]$.

Proof. One applies Proposition 3.2 and the mean value theorem for univariate and respectively bivariate divided differences. □

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