CREATIVE MATH. & INF.
17 (2008), 8 - 13

About Simpson-type and Hermite-type inequalities

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ABSTRACT. In this paper we introduce new means and prove new relations between classical means based on the Simpson formula and Hermite integral inequality.

1. INTRODUCTION

Let be $0 < a \leq b$, then $A(a,b) = \frac{a+b}{2}$ is the arithmetical mean, $G(a,b) = \sqrt{ab}$ is the geometrical mean, $H(a,b) = \frac{2ab}{a+b}$ is the harmonical mean,

$$L(a,b) = \frac{b-a}{\ln b - \ln a}$$

is the logarithmic mean, $I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ is the identric mean.

In [2], M. Bencze introduced a method to obtain new means and refinements. In this paper we apply this method to Simpson formula and to Hermite integral inequality, to obtain new relations and refinements between classical means.

2. SIMPSON-TYPE INEQUALITIES

Theorem 2.1. *If $f : [a,b] \to R$ is four times differentiable and $f^{(4)}(x) \geq 0$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \leq \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$. If $f^{(4)}(x) < 0$ for all $x \in [a,b]$, then the reverse inequality holds.*

Proof. Using the Simpson formula we have

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) - \frac{(b-a)^5}{2880} f^{(4)}(s),$$

where $s \in [a,b]$. If $f^{(4)}(x) > 0$, for all $x \in [a,b]$, then we have the first inequality, if $f^{(4)}(x) < 0$ for all $x \in [a,b]$, then we obtain the reverse inequality. Equality in Simpson formula is obtained for polynomials of degree three. □

Received: 15.10.2007. In revised form: 11.11.2007.
2000 *Mathematics Subject Classification.* 26D15.
Key words and phrases. *Hermite-Hadamarad-Jensen's integral inequality, Simpson formula, convex functions.*

Corollary 2.1. *If* $\alpha \in (-\infty, 0) \cup (1,2) \cup (3, +\infty)$, *then*

$$\frac{L\left(a^{\alpha+1}, b^{\alpha+1}\right)}{L(a,b)} \le \frac{1}{3} A\left(a^{\alpha}, b^{\alpha}\right) + \frac{2}{3} A^{\alpha}(a,b).$$

If $\alpha \in (0,1) \cup (2,3)$, *then we have the reverse inequality.*

Proof. In Theorem 2.1, we take $f(x) = x^{\alpha}$. □

Problem 2.1. Determine the best constants $x, y > 0$, such that

$$(x+y)\frac{L\left(a^{\alpha+1}, b^{\alpha+1}\right)}{L(a,b)} \le xA\left(a^{\alpha}, b^{\alpha}\right) + yA^{\alpha}(a,b)$$

for all $a, b > 0$ and $\alpha \in (-\infty, 0) \cup (1,2) \cup (3, +\infty)$.

Corollary 2.2. *If* $\alpha \ge 0$, *then*

$$\frac{3}{L(a+\alpha, b+\alpha)} \le \frac{1}{H(a+\alpha, b+\alpha)} + \frac{2}{A(a+\alpha, b+\alpha)}.$$

Proof. In Theorem 2.1, we take $f(x) = \dfrac{1}{x+\alpha}$. □

Problem 2.2. Determine the best constants $x, y > 0$, such that

$$\frac{x+y}{L(a+\alpha, b+\alpha)} \le \frac{x}{H(a+\alpha, b+\alpha)} + \frac{y}{A(a+\alpha, b+\alpha)},$$

for all $a, b > 0$ and $\alpha \ge 0$.

Corollary 2.3. *If* $\alpha \in R$, *then* $3L\left(a^{\alpha}, b^{\alpha}\right) \le A\left(a^{\alpha}, b^{\alpha}\right) + 2G^{\alpha}(a,b)$.

Proof. In Theorem 2.1, we take $f(x) = e^{\alpha x}$ and $a \to \ln a$, $b \to \ln b$. □

Problem 2.3. Determine the best constants $x, y > 0$, such that

$$(x+y) L\left(a^{\alpha}, b^{\alpha}\right) \le xA\left(a^{\alpha}, b^{\alpha}\right) + yG^{\alpha}(a,b)$$

for all $x \in R$ and $a, b > 0$.

Corollary 2.4. *If* $\alpha \ge 0$, *then*

$$I^{3}(a+\alpha, b+\alpha) \ge A^{2}(a+\alpha, b+\alpha)\, G(a+\alpha, b+\alpha).$$

Proof. In Theorem 2.1, we take $f(x) = \ln(x+\alpha)$. □

Problem 2.4. Determine the best constants $x, y > 0$, such that

$$I^{x+y}(a+\alpha, b+\alpha) \ge A^{x}(a+\alpha, b+\alpha)\, G^{y}(a+\alpha, b+\alpha)$$

for all $\alpha \ge 0$ and $a, b > 0$.

In the following we give some refinements for the proved inequalities.

Theorem 2.2. *If* $f : [a,b] \to R$ *is absolute continuous, differentiable and* $m_1 \le f'(x) \le m_2$, *for all* $x \in [a,b]$, *then*

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right) \right| \le \frac{5}{72}(m_2 - m_1)(b-a)^2.$$

Proof. If $P(x)=\begin{cases} x-\dfrac{5a+b}{6}, & \text{if } x\in\left[a,\dfrac{a+b}{2}\right] \\ x-\dfrac{a+5b}{6}, & \text{if } x\in\left[\dfrac{a+b}{2},a\right]\end{cases}$, then

$$\int_a^b P(x)f'(x)\,dx = \int_a^{\frac{a+b}{2}}\left(x-\frac{5a+b}{6}\right)f'(x)\,dx+\int_{\frac{a+b}{2}}^b\left(x-\frac{a+5b}{6}\right)f'(x)\,dx = \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\int_a^b f(x)\,dx$$

and

$$\int_a^b |P(x)|\,dx = \int_a^{\frac{a+b}{2}}\left|x-\frac{5a+b}{6}\right|dx+\int_{\frac{a+b}{2}}^b\left|x-\frac{a+5b}{6}\right|dx=\frac{5}{6}(b-a)^2,$$

$$\int_a^b P(x)\,dx = 0, \int_a^b P(x)f'(x)\,dx=\int_a^b P(x)\left(f'(x)-\frac{m_1+m_2}{2}\right)dx,$$

therefore

$$\left|\int_a^b f(x)\,dx-\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right|=$$
$$=\left|\int_a^b P(x)f'(x)\,dx\right|=\left|\int_a^b P(x)\left(f'(x)-\frac{m_1+m_2}{2}\right)dx\right|\le$$
$$\le\max\left|f'(x)-\frac{m_1+m_2}{2}\right|\int_a^b|P(x)|\,dx\le\frac{m_2-m_1}{2}\int_a^b|P(x)|\,dx=$$
$$=\frac{5}{72}(m_2-m_1)(b-a)^2.$$

□

Corollary 2.5. *If* $\alpha\in(1,2)\cup(3,+\infty)$, *then*

$$A(a^\alpha,b^\alpha)+2A^\alpha(a,b)-3L\left(a^{\alpha+1},b^{\alpha+1}\right)\le$$
$$\le\frac{5\alpha}{24}\left(A(a^\alpha,b^\alpha)-G^2(a,b)A\left(a^{\alpha-2},b^{\alpha-2}\right)\right)$$

and if $\alpha\in(0,1)\cup(2,3)$, *then*

$$3L\left(a^{\alpha+1},b^{\alpha+1}\right)-A(a^\alpha,b^\alpha)-2A^\alpha(a,b)\le$$
$$\le\frac{5\alpha}{24}\left(A(a^\alpha,b^\alpha)-G^2(a,b)A\left(a^{\alpha-2},b^{\alpha-2}\right)\right).$$

Proof. In Theorem 2.2 we take $f(x)=x^\alpha$. □

Corollary 2.6. *If* $\alpha \geq 0$, *then*

$$\frac{3}{L(a+\alpha,b+\alpha)} - \frac{1}{H(a+\alpha,b+\alpha)} - \frac{2}{A(a+\alpha,b+\alpha)} \leq$$

$$\leq \frac{5A(a+\alpha,b+\alpha)\left(A\left(a^3,b^3\right) - G(a,b)A(a,b)\right)}{3G^4(a,b)}.$$

Proof. In Theorem 2.2 we take $f(x) = \frac{1}{x+\alpha}$. □

Corollary 2.7. *If* $\alpha > 0$, *then*

$$A(a^\alpha,b^\alpha) + 2G^\alpha(a,b) - 3L(a^\alpha,b^\alpha) \leq$$

$$\leq \frac{5\alpha\left(A\left(a^{\alpha+1},b^{\alpha+1}\right) - G^2(a,b)A\left(a^{\alpha-1},b^{\alpha-1}\right)\right)}{12L(a,b)}.$$

Proof. In Theorem 2.2 we take $f(x) = e^{\alpha x}$, $a \to \ln a$, $b \to \ln b$. □

Corollary 2.8. *If* $\alpha \geq 0$, *then*

$$\frac{I^3(a+\alpha,b+\alpha)}{G(a+\alpha,b+\alpha)A^2(a+\alpha,b+\alpha)} \leq \exp\left(\frac{5\left(A\left(a^2,b^2\right) - G^2(a,b)\right)}{6G^2(a+\alpha,b+\alpha)}\right).$$

Proof. In Theorem 2.2 we take $f(x) = \ln(x+\alpha)$. □

3. Hermite-type inequalities

Theorem 3.1. *Let* $f:[a,b] \to R$ *be a nonconstant convex function,* $M = \sup_{x\in(a,b)} |f'(x)|$, *then*

$$\max\left\{(b-a)f\left(\frac{a+b}{2}\right);\right.$$

$$\left.\frac{1}{4M}(f(b)-f(a))^2 - \frac{M}{4}(b-a)^2 + \frac{1}{2}(b-a)(f(a)+f(b))\right\} \leq$$

$$\leq \int_a^b f(x)\,dx \leq min\left\{\frac{1}{2}(b-a)(f(a)+f(b));\right.$$

$$\left.\frac{1}{2}(b-a)(f(a)+f(b)) + \frac{M}{4}(b-a)^2 - \frac{1}{4M}(f(b)-f(a))^2\right\}$$

Proof. We have the following inequalities:

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)\,dx \leq \frac{1}{2}(b-a)(f(a)+f(b))$$

(see [3]), and

$$\frac{1}{4M}\left(f\left(b\right)-f\left(a\right)\right)^2-\frac{M}{4}\left(b-a\right)^2+\frac{1}{2}\left(b-a\right)\left(f\left(a\right)+f\left(b\right)\right)\leq$$

$$\leq\int_a^b f\left(x\right)dx\leq\frac{1}{2}\left(b-a\right)\left(f\left(a\right)+f\left(b\right)\right)+\frac{M}{4}\left(b-a\right)^2-\frac{1}{4M}\left(f\left(b\right)-f\left(a\right)\right)^2$$

(see [5]). □

Corollary 3.1. *If* $\alpha\in(-\infty,0)\cup(1,2)\cup(3,+\infty)$ *then*

$$\max\left\{A^{\alpha}\left(a,b\right);\frac{A\left(a^{2\alpha},b^{2\alpha}\right)-G^{\alpha}\left(a,b\right)}{2\sqrt{2}\alpha b^{\alpha-1}\sqrt{A\left(a^2,b^2\right)-G^2\left(a,b\right)}}-\right.$$

$$\left.-\frac{\alpha b^{\alpha-1}\sqrt{A\left(a^2,b^2\right)-G^2\left(a,b\right)}}{2\sqrt{2}}+A\left(a^{\alpha},b^{\alpha}\right)\right\}\leq\frac{L\left(a^{\alpha+1},b^{\alpha+1}\right)}{L\left(a,b\right)}\leq$$

$$\leq\min\left\{A\left(a^{\alpha},b^{\alpha}\right);A\left(a^{\alpha},b^{\alpha}\right)+\frac{\alpha b^{\alpha-1}\sqrt{A\left(a^2,b^2\right)-G^2\left(a,b\right)}}{2\sqrt{2}}-\right.$$

$$\left.-\frac{A\left(a^{2\alpha},b^{2\alpha}\right)-G^{\alpha}\left(a,b\right)}{2\sqrt{2}\alpha b^{\alpha-1}\sqrt{A\left(a^2,b^2\right)-G^2\left(a,b\right)}}\right\}$$

Proof. In Theorem 3.1, we take $f\left(x\right)=x^{\alpha}$. □

Corollary 3.2. *If* $\alpha\geq 0$, *then*

$$\max\left\{\frac{1}{A\left(a,b\right)+\alpha};\frac{\left(a+\alpha\right)^2\sqrt{A\left(a^2,b^2\right)-G^2\left(a,b\right)}}{2\sqrt{2}G^4\left(a+\alpha,b+\alpha\right)}-\right.$$

$$\left.-\frac{\sqrt{A\left(a^2,b^2\right)-G^2\left(a,b\right)}}{2\sqrt{2}\left(a+\alpha\right)^2}+\frac{A\left(a,b\right)+\alpha}{G^2\left(a+\alpha,b+\alpha\right)}\right\}\leq\frac{1}{L\left(a+\alpha,b+\alpha\right)}\leq$$

$$\leq\min\left\{\frac{A\left(a,b\right)+\alpha}{G^2\left(a+\alpha,b+\alpha\right)};\frac{A\left(a,b\right)+\alpha}{G^2\left(a+\alpha,b+\alpha\right)}+\frac{\sqrt{A\left(a^2,b^2\right)-G^2\left(a,b\right)}}{2\sqrt{2}\left(a+\alpha\right)^2}-\right.$$

$$\left.-\frac{\left(a+\alpha\right)^2\sqrt{A\left(a^2,b^2\right)-G^2\left(a,b\right)}}{2\sqrt{2}G^4\left(a+\alpha,b+\alpha\right)}\right\}.$$

Proof. In Theorem 3.1, we take $f\left(x\right)=\frac{1}{x+\alpha}$. □

Corollary 3.3. *If* $\alpha>0$, *then*

$$\max\left\{G^{\alpha}\left(a,b\right);\frac{L\left(a,b\right)\left(A\left(a^{\alpha},b^{\alpha}\right)-G^{\alpha}\left(a,b\right)\right)}{2\sqrt{2}\alpha b^{\alpha}\sqrt{A\left(a^2,b^2\right)-G^2\left(a,b\right)}}-\right.$$

$$\left.-\frac{\alpha b^{\alpha}\sqrt{A\left(a^2,b^2\right)-G^2\left(a,b\right)}}{2\sqrt{2}L\left(a,b\right)}+A\left(a^{\alpha},b^{\alpha}\right)\right\}\leq L\left(a^{\alpha},b^{\alpha}\right)\leq$$

$$\leq \min\left\{A\left(a^{\alpha},b^{\alpha}\right);A\left(a^{\alpha},b^{\alpha}\right)+\frac{\alpha b^{\alpha}\sqrt{A\left(a^{2},b^{2}\right)-G^{2}\left(a,b\right)}}{2\sqrt{2}L\left(a,b\right)}-\right.$$
$$\left.-\frac{L\left(a,b\right)\left(A\left(a^{\alpha},b^{\alpha}\right)-G^{\alpha}\left(a,b\right)\right)}{2\sqrt{2}b^{\alpha}\sqrt{A\left(a^{2},b^{2}\right)-G^{2}\left(a,b\right)}}\right\}.$$

Proof. In Theorem 3.1, we take $f\left(x\right)=e^{\alpha x}$ and $a\rightarrow\ln a$, $b\rightarrow\ln b$. □

Corollary 3.4. *If* $\alpha\geq 0$, *then*

$$\max\left\{\frac{1}{2}\ln\left(G^{2}\left(a,b\right)+2\alpha A\left(a,b\right)+\alpha^{2}\right);\frac{1}{2}\ln\left(G^{2}\left(a,b\right)+2\alpha A\left(a,b\right)+\alpha^{2}\right)-\right.$$
$$\left.-\frac{\sqrt{A\left(a^{2},b^{2}\right)-G^{2}\left(a,b\right)}}{2\sqrt{2}\left(a+\alpha\right)}+\frac{\left(a+\alpha\right)\sqrt{A\left(a^{2},b^{2}\right)-G^{2}\left(a,b\right)}}{2\sqrt{2}L^{2}\left(a+\alpha,b+\alpha\right)}\right\}\leq$$
$$\leq\ln I\left(a+\alpha,b+\alpha\right)\leq$$
$$\leq\min\left\{\ln\left(A\left(a,b\right)+\alpha\right);\frac{1}{2}\ln\left(G^{2}\left(a,b\right)+2\alpha A\left(a,b\right)+\alpha^{2}\right)+\right.$$
$$\left.+\frac{\sqrt{A\left(a^{2},b^{2}\right)-G^{2}\left(a,b\right)}}{2\sqrt{2}\left(a+\alpha\right)}-\frac{\left(a+\alpha\right)\sqrt{A\left(a^{2},b^{2}\right)-G^{2}\left(a,b\right)}}{2\sqrt{2}L^{2}\left(a+\alpha,b+\alpha\right)}\right\}$$

Proof. In Theorem 3.1, we take $f\left(x\right)=-\ln\left(x+\alpha\right)$. □

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