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Theorems of the type of Cutler for abelian p-groups

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ABSTRACT. Suppose G and H are abelian p-groups. It is shown that if G and H are quasi-isomorphic then G is (a) summable or (b) σ-summable or (c) $p^{\omega+m}$-projective, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ or (d) a strong ω-elongation of a totally projective (respectively summable) group by a $p^{\omega+m}$-projective group, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ or (e) thick if and only if so is H. These five independent claims complemented results of this type due to Cutler (appeared in Pac. J. Math., 1966) and are supplements to our recent results (published in Proc. Indian Acad. Sci.-Math. Sci., 2004) too.

1. INTRODUCTION

In the fundamental paper [3] (see [9] as well), Cutler states, not in an explicit form however, the following important question.

Problem: Let G be an abelian p-group and let S be its subgroup such that $p^nG \subseteq S \subseteq G$ for some arbitrary but a fixed positive integer n. Does it follow then that $G \in \mathcal{K}$ if and only if $S \in \mathcal{K}$, provided $\mathcal{K}$ is a class of abelian p-groups?

We emphasize that each positive result of this kind will be hereafter termed as *a theorem of the type of Cutler* for the concrete class of abelian p-groups.

Cutler answers in the affirmative the posed problem for the classes of: (1) direct sums of cyclic p-groups; (2) closed p-groups and (3) Σ-p-groups. He also asks whether this is the case for direct sums of countable abelian p-groups and direct sums of closed p-groups. Irwin and Richman have positively settled in [9] the first query, while the second has a denial answer in [8]. Following the idea of Irwin-Richman and some principal known facts the assertion will be extended in the sequel to totally projective p-groups and to some natural generalizations of this group sort. Besides, Richman shows in ([12], Theorem 1) that the same holds true for an independent class of groups termed by him as thin groups. In this aspect, in ([9], Proposition 3, Properties 3. and 5.) was also established a satisfactory solution of the foregoing question for the class of so-called Q-groups. On the other hand, in ([6], Corollary 4.5) Eklof jointly with Huber obtained that an abelian p-group G is weakly ω_1-separable $\iff$ p^nG is weakly ω_1-separable. Since it is well-known that a subgroup of a weakly ω_1-separable group is again weakly ω_1-separable (see the criterion due to Megibben listed below), a corresponding theorem of the type of Nunke for weakly ω_1-separable groups is also fulfilled. Nevertheless, we shall give in the next lines an independent confirmation of this fact.

The Cutler's statement corresponds with the problems of the kind of Nunke (see, for instance, [4]) and with the description of the structure of large subgroups (see, for example, [1] and [5]). In fact, for any basic subgroup B of G we derive $G = (p^nG)B = SB$. However, S is not equal to L, a large subgroup of G, since S

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may not be fully invariant (= completely characteristic) in G. Moreover, it is well-known that G/L is a direct sum of cyclic p-groups (see e.g. [1] or [7]) whereas in our current situation G/S must be bounded.

2. Quasi-isomorphism for primary abelian groups

The section is concerned with properties of p-primary abelian groups that are invariant under the relation of quasi-isomorphism. Imitating [3], two abelian p-groups G and H are quasi-isomorphic if and only if there exist positive integers m and n and subgroups K and L of G and H, respectively, such that $p^nG \subset K, p^mH \subset L$ and $K \cong L$. As usual, the p^α-powers of G are defined inductively as follows: $p^\alpha G = p(p^{\alpha-1}G)$ if α is non-limit, i.e., $\alpha - 1$ exists, or $p^\alpha G = \cap_{\beta<\alpha} p^\beta G$ otherwise. For a simplicity in some records, we denote $p^\omega G$ by G^1. All other notions and notations being standard are the same as in [7].

The aim of the present article is to answer the stated above Problem due to Cutler in some partial cases and, in particular, the question posed also in [3], namely what does quasi-isomorphism have to say about abelian p-groups?

The definitions of the investigated classical sorts of primary groups such as summable groups and totally projective groups can be found in [7]. Nevertheless, for completeness of the exposition, we shall remember the first of them as well as we shall recollect some criteria for attractive classes of groups. In fact, an abelian p-group G is said to be summable if $G[p] = \oplus_{\alpha<\lambda} S_\alpha$, where, for each $\alpha < \lambda$, $S_\alpha \setminus \{0\} \subseteq p^\alpha G \setminus p^{\alpha+1}G$; $\lambda = \text{length}(G)$. Moreover, an abelian p-group G is called σ-summable only when $G[p] = \cup_{k<\omega} G_k, G_k \subseteq G_{k+1}$ and, $\forall\, k \geq 1$, $\exists\, \alpha_k < \text{length}(G)$: $G_k \cap p^{\alpha_k}G = 0$. Finally, an abelian p-group G is said to be a strong ω-elongation of a totally projective (respectively summable) group by a $p^{\omega+m}$-projective group precisely when $p^\omega G$ is totally projective (respectively summable) and $\exists\, P \leq G[p^m]$: $G/(P + p^\omega G)$ is a direct sum of cyclics. It is clear that for such a group G we have that $p^\omega G$ is totally projective and $G/p^\omega G$ is $p^{\omega+m}$-projective, while the converse implication fails to be ever true.

In addition, the following three criteria are of interest.

Criterion ([11]). *An abelian p-group G is $p^{\omega+m}$-projective for $m \in \mathbb{N}_0 \iff \exists\, M \leq G[p^m]$ so that G/M is a direct sum of cyclic groups.*

Criterion ([2], Theorem 3.2 - (3)). *An abelian p-group G is thick $\iff C \supset (p^dG)[p]$ for some $d \in \mathbb{N}$ and for every $C \leq G$ such that G/C is a direct sum of cyclics.*

Criterion ([10], Theorem 1). *A separable abelian p-group G is weakly ω_1-separable $\iff \forall\, C \leq G$: $|C| = \aleph_0 \Rightarrow |\cap_{i<\omega} (C + p^iG)| = \aleph_0 \iff \forall\, C \leq G$: $|C| = \aleph_0 \Rightarrow |(G/C)^1| \leq \aleph_0$.*

Referring to [9], a separable abelian p-group G is a Q-group if $|(G/C)^1| \leq |C|$ whenever $C \leq G$ with $|C| \geq \aleph_0$. Thus it is easily seen that any Q-group is weakly ω_1-separable, while each weakly ω_1-separable group of cardinality not exceeding $\aleph_1$ is a Q-group; the limitation on the power cannot be dropped off. These two implications are consistent in (ZFC) and are independent from the other additional axioms of the set theory.

All we have to do is to check the validity of the following main attainment.

Theorem 2.1. *Suppose G is an abelian p-group with a subgroup S so that $p^nG \subseteq S$ for some nonnegative integer n. Then S has one of the following properties*

(a) summable;

(b) σ-summable;

(c) $p^{\omega+m}$-projective, $m \in \mathbb{N}_0$;

(d) a strong ω-elongation of a totally projective (respectively summable) group by a $p^{\omega+m}$-projective group, $m \in \mathbb{N}_0$;

(e) thick

if and only if G has the same property.

Proof. (a) Assume G is summable. Hence both p^nG and $p^\omega G = p^\omega S$ are summable as special fully invariant subgroups (see cf. [5] too). On the other hand, G and p^nG being summable groups ensure that they are Σ-groups (see, for example, [7]). In virtue of [3], S must be a Σ-group. Now, we shall deduce that S is summable. Indeed, denote by H_S some arbitrary high subgroup of S. Thus $S[p] = H_S[p] \oplus p^\omega S[p]$. Moreover, H_S is a direct sum of cyclics. Furthermore, we may write $H_S[p] = \oplus_{k<\omega} H_k$, where, for each $k < \omega$, $H_k \setminus \{0\} \subseteq p^kS \setminus p^{k+1}S$ because of the fact that H_S is pure in S (see, for instance, [7]). The summable $p^\omega S$ implies $p^\omega S[p] = \oplus_{\alpha<\lambda} S_\alpha$, where, for each $\alpha < \lambda$, $S_\alpha \setminus \{0\} \subseteq p^{\omega+\alpha}S \setminus p^{\omega+\alpha+1}S$. Consequently, $S[p] = \oplus_{k<\omega} H_k \oplus \oplus_{\alpha<\lambda} S_\alpha = \oplus_{\beta<\omega+\lambda} K_\beta$ by putting $K_k = H_k$ whenever $k < \omega$ and $K_{\omega+\alpha} = S_\alpha$ whenever $0 \leq \alpha < \lambda$. Besides, we obviously calculate that $K_\beta \setminus \{0\} \subseteq p^\beta S \setminus p^{\beta+1}S$, for all $\beta < \omega + \lambda$. Finally, we infer that S is summable, thus completing the first half.

Conversely, we presume now that S is summable. Hence S is a Σ-group and so in conjunction with [3], G is a Σ-group. Besides S being summable yields that $p^\omega S = p^\omega G$ is summable. By the method what we have just illustrate above, G would be summable as desired. The proof of the first point is completed after all.

Let us now consider (b). Foremost, given that G is σ-summable. By definition we can write $G[p] = \cup_{k<\omega} G_k$, $G_k \subseteq G_{k+1}$ and for every natural number k there is an ordinal α_k with the property $G_k \cap p^{\alpha_k}G = 0$ and $\alpha_k <$ length(G). Therefore, we obtain $S[p] = \cup_{k<\omega}(G_k \cap S)$ and $G_k \cap S \subseteq G_{k+1} \cap S$. Next, we compute that length$(S) =$ length(G) whenever length$(G) \geq \omega$. This is so because $p^\omega G = p^\omega S$. Furthermore, $G_k \cap S \cap p^{\alpha_k}S \subseteq G_k \cap p^{\alpha_k}G = 0$ where $\alpha_k <$ length(S), and we are done. If now G is bounded, it is clear that so is S and oppositely, thus there is nothing to prove.

For the converse, we write down $S[p] = \cup_{k<\omega} S_k$, $S_k \subseteq S_{k+1}$ and $S_k \cap p^{\alpha_k}S = 0$ for all $k \geq 0$ and some $\alpha_k <$ length$(S) =$ length$(G) \geq \omega$; if length$(G) < \omega$, we are finished. We obviously observe that $(p^nG)[p] = \cup_{k<\omega}(S_k \cap p^nG)$, $S_k \cap p^nG \subseteq S_{k+1} \cap p^nG$ and $S_k \cap p^nG \cap p^{n+\alpha_k}G = S_k \cap p^{n+\alpha_k}G \subseteq S_k \cap p^{\alpha_k}S = 0$ where $n + \alpha_k <$ length(G) since length(G) is limit being co-final with ω. That is why p^nG is σ-summable. Bearing in mind [4], G must be σ-summable as well, as expected. This completes the second point.

(c) If G is $p^{\omega+m}$-projective, then the same holds valid for S as its subgroup (see cf. [11]).

Now, we treat the more difficult reverse question. So, given that S is $p^{\omega+m}$-projective. Applying the foregoing criterion from [11], there exists a subgroup $E \leq S[p^m]$ with the property S/E is a direct sum of cyclics. By virtue of a result due to L. Kulikov (see, for instance, cf. [7]), the subgroup $(p^nG+E)/E \subseteq S/E$ is a direct sum of cyclic groups. Taking into account that $p^n(G/E) = (p^nG+E)/E$, the application of ([7], vol. I, p. 111, Proposition 18.3) leads us to G/E is a direct sum of cyclics. Since $E \leq G[p^m]$, employing [11], G should be $p^{\omega+m}$-projective as promised.

(d) First of all, let G be a strong ω-elongation of a totally projective group by a $p^{\omega+m}$-projective group. By definition $p^\omega G$ is totally projective and there is $P \leq G[p^m]$ so that $G/(P+p^\omega G)$ is a direct sum of cyclics. Since $p^nG \subseteq S$, it follows that $p^\omega G = p^\omega S$, hence $p^\omega S$ is totally projective. On the other hand, with the aid of the modular law from [7], $G/(P+p^\omega G) \supseteq (S+P)/(P+p^\omega G) = (S+P+p^\omega S)/(P+p^\omega S) \cong S/(S\cap(P+p^\omega S)) = S/(p^\omega S+(S\cap P))$ is a direct sum of cyclics being a subgroup of a direct sum of cyclics (see cf. [7]). Since $S\cap P \subseteq S[p^n]$ we are finished.

Reciprocally, let S be a strong ω-elongation of a totally projective group by a $p^{\omega+m}$-projective group. As in the preceding consideration $p^\omega S = p^\omega G$ is totally projective. Moreover, if $T \subseteq S[p^n]$ with $S/(T+p^\omega S)$ a direct sum of cyclic groups, we observe that $G/(T+p^\omega S)/S/(T+p^\omega S) \cong G/S$ is bounded and consequently, by virtue of ([7], vol. I, p. 111, Proposition 18.3), we deduce that $G/(T+p^\omega S) = G/(T+p^\omega G)$ is a direct sum of cyclics, where $T \subseteq G[p^n]$. The situation for summable groups is analogous. So, we are done.

(e) Firstly, given that G is thick and S/M is a direct sum of cyclics for some arbitrary subgroup M of S. Since $G/M/S/M \cong G/S$ is bounded, we conclude as above that G/M is a direct sum of cyclic groups. The corresponding criterion, stated above, allows us to write that there exists $d \geq 1$ such that $(p^dG)[p] \subset M$. Hence $(p^dS)[p] \subset M$, thus S is thick and this completes the sufficiency.

Turning to the opposite part-half, given S is thick with G/C a direct sum of cyclics for some $C \leq G$. Furthermore, by [7], $(S+C)/C \subseteq G/C$ is also a direct sum of cyclic groups and that is why the Criterion enables us to infer that the inclusion $(p^d(S+C))[p] = (p^dS+p^dC)[p] \subset C$; thus $(p^dS)[p] \subset C$. But, because $p^nG \subseteq S$, we plainly obtain that $(p^{d+n}G)[p] \subset C$, which is enough to say that G is thick, as expected.

The proof of the theorem is finished after all. □

The previous assertion of the theorem can equivalently be restated like this.

Proposition 2.1. *Let G be an abelian p-group with $S \leq G$ such that G/S is bounded. Then G is either*

(a) summable;

(b) σ-summable;

(c) $p^{\omega+m}$-projective, $m \in \mathbb{N}_0$;

(d) a strong ω-elongation of a totally projective (respectively summable) group by a $p^{\omega+m}$-projective group, $m \in \mathbb{N}_0$;

(e) thick

if and only if so is S.

The following consequence is important.

Corollary 2.1. *Let the abelian p-groups G and H be quasi-isomorphic. If G belongs to the classes of groups listed in the Theorem, then so does H.*

Proof. According to the definition for quasi-isomorphism and utilizing the method described in ([3], Proposition 3.3) combined with our Theorem, we are done. □

We shall now give an easy alternative confirmation only for the primary case of the alluded to above result from ([9], p. 446, Corollary 3) which solves the Cutler's problem for direct sums of countable abelian p-groups.

Claim (Irwin-Richman). *Suppose $G \geq S$ is an abelian p-group so that G/S is bounded. Then G is a direct sum of countable groups $\iff$ S is a direct sum of countable groups.*

Proof. "$\Rightarrow$". Evidently, there is $n \in \mathbb{N}$ such that $p^nG \subseteq S$, whence $p^\omega G = p^\omega S$ is a direct sum of countable groups (see, for instance, [7]). Moreover, it is well-known that $G/p^\omega G$ is a direct sum of cyclic groups (cf. [7]). Therefore, $S/p^\omega S = S/p^\omega G \subseteq G/p^\omega G$ has the same property. Finally, again appealing to [7], we conclude that S possesses the desired decomposition into countable factors.

"$\Leftarrow$". By the same token as in the previous point, $p^\omega S = p^\omega G$ is a direct sum of countable groups and $S/p^\omega G$ is a direct sum of cyclics. As above, what we need to show is that $G/p^\omega G$ is a direct sum of cyclics as well. Here we can give two independent approaches. Firstly, since $G/p^\omega G/S/p^\omega G \cong G/S$ is bounded, the claim follows from ([7], vol. I, p. 111, Proposition 18.3). Secondly, since $S/p^\omega G = \cup_{i<\omega}(S_i/p^\omega G)$ with $S_i \cap p^iS \subseteq p^\omega G\ \forall\ i \geq 1$ and since there exists $n \in \mathbb{N}$ with $p^nG \subseteq S$, we subsequently deduce for each $i \geq 1$ that $S_i \cap p^{n+i}G \subseteq S_i \cap p^iS \subseteq p^\omega G$, hence the elements of every $S_i/p^\omega G$ has heights bounded in general as computed in $G/p^\omega G$. We now wish to apply the Dieudonné criterion (see, for example, [7] or [5]) to infer the wanted property for $G/p^\omega G$. □

As aforementioned, the same claim holds true for totally projective groups (e.g. [7]), the proof of which is similar.

We shall also give the promised above new verification of the truthfulness of the Cutler's type theorem for weakly ω_1-separable p-groups, established by Eklof and Huber ([6], Corollary 4.5).

Claim (Eklof-Huber). *Suppose $G \geq S$ is an abelian p-group so that G/S is bounded. Then G is weakly ω_1-separable $\iff$ S is weakly ω_1-separable.*

Proof. By hypothesis, there is $n \in \mathbb{N}$ with the property $p^nG \subseteq S$, whence $p^\omega G = p^\omega S$. Thereby, G is separable precisely when so is S.

Moreover, it is straightforward to look at the corresponding Criterion that any subgroup of a weakly ω_1-separable group is weakly ω_1-separable too. Thus if G is weakly ω_1-separable, then so does S.

Conversely, if S is weakly ω_1-separable, then by what we have already observed the same property follows for p^nG. To show that G is weakly ω_1-separable, given a countable subgroup C of G. The showing is accomplished with the checking that $|\cap_{i<\omega}(C + p^{n+i}G)| = \aleph_0$. In fact, in doing this, we consider two possible cases.

Case 1: $p^nG \cap C$ is finite. Therefore, it is nice in p^nG and thereby $p^nG/(C \cap p^nG) \cong (p^nG + C)/C$ is separable because so is p^nG. Besides, $[\cap_{i<\omega}(C + p^{n+i}G)]/C = \cap_{i<\omega}((C+p^{n+i}G)/C) = \cap_{i<\omega}p^i((p^nG+C)/C) = ((p^nG+C)/C)^1 = 0$. Thus $\cap_{i<\omega}(C + p^{n+i}G) = C$ and therefore $|\cap_{i<\omega}(C + p^{n+i}G)| = |C| = \aleph_0$.

Case 2: $p^nG \cap C$ is countably infinite. Since S is weakly ω_1-separable, as aforementioned so is p^nG being its subgroup and thus an appeal to the foregoing Criterion assures that $(p^nG/(p^nG \cap C))^1 \cong ((C + p^nG)/C)^1 = [\cap_{i<\omega}(C + p^{n+i}G)]/C$ is at most countable. Consequently $| \cap_{i<\omega} (C + p^{n+i}G)| = |C| = \aleph_0$.

Finally, in both cases, we have $\cap_{i<\omega}(C + p^iG) = \cap_{i<\omega}(C + p^{n+i}G)$ and thus $| \cap_{i<\omega} (C + p^iG)| = | \cap_{i<\omega} (C + p^{n+i}G)| = \aleph_0$, as required. That is why, G is weakly ω_1-separable as well. □

In closing, we shall confirm once again that the Cutler's problem possesses a negative answer in general (see [8] as well); specifically we construct the following example that is our goal here: An abelian p-group is said to be starred if it has the same power as its basic subgroup. Suppose S contains p^nG and set $p^kG = S$ for any $k < n$ and $n \geq 2$. Given now that G is unbounded torsion-complete with a basic subgroup $B = \oplus_{0<s\leq k} \oplus_{\aleph_1} Z(p^s) \oplus \oplus_{k<s<\omega} \oplus_{\aleph_0} Z(p^s)$. Evidently B is of cardinality $\aleph_1$ whereas p^kB is of cardinality $\aleph_0$. Also, we assume that the *Generalized Continuum Hypothesis* holds. Henceforth, we claim that G is starred and, by consulting with [3], that p^kG is unbounded torsion-complete but not starred. This is true because p^kB is a basic subgroup of p^kG (see [7]), and thus by making use of ([7], p. 29, Exercise 7) we compute $|G| = |B|^{\aleph_o} = \aleph_1^{\aleph_o} = \aleph_o^{\aleph_o} = \aleph_1 = |B| = |p^kG| = |p^kB|^{\aleph_o} > \aleph_o = |p^kB|$, that gives the claim. Notice that if p^kG is starred for some $k \in \mathbb{N}$, then G is also starred (e.g. [9], p. 446, Proposition 1).

It is not hard to verify in general that $|G| > |S|$ even when $\exists\, n \in \mathbb{N}$ with $S \supseteq p^nG$, so S countable does not imply that the same is G.

We terminate the investigation with some queries which are left-open yet.

3. Open problems

Here we quote some conjectures which remain still unsolved. Whether or not the Cutler's question has a positive solution for the classes of: (1) direct sums of σ-summable abelian p-groups and (2) essentially finitely indecomposable p-groups (for the definition of these groups we refer the reader to [2]). We conjecture that the answer is no for the first situation but yes for the second one. In the cases when G is of the class of: (3) pure-complete p-groups; (4) semi-complete p-groups; (5) direct sums of torsion-complete p-groups (a problem from [3]), we once again remember that the negative settling of the posed Cutler's query was obtained by P. Hill in [8, Theorem 4.1].

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