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A formula involving the Bernstein fundamental polynomials

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ABSTRACT. In this article we want to determinate a relation between the Bernstein fundamental multivariate polinomyals.

1. PRELIMINARIES

In this section we recall some notions and results which we will use in this paper. Let $B_m : C(\Delta_k) \to C(\Delta_k)$, m a non zero natural number, be the Bernstein multivariate operators (see [2]) defined for any function $f \in C(\Delta_k)$ by

$$(B_m f)(x_1, \ldots, x_k) = \sum_{\substack{\nu_1, \ldots, \nu_k \geq 0 \\ \nu_1 + \ldots + \nu_k \leq m}} p_{\nu_1, \ldots, \nu_k; m}(x_1, \ldots, x_k) \cdot f\left(\frac{\nu_1}{m}, \ldots, \frac{\nu_k}{m}\right) \tag{1.1}$$

where $p_{\nu_1, \ldots, \nu_k; m}(x_1, \ldots, x_k)$ are the fundamental polynomials, defined by

$$p_{\nu_1, \ldots, \nu_k; m}(x_1, \ldots, x_k) = \binom{m}{\nu_1, \ldots, \nu_k} x_1^{\nu_1} \ldots x_k^{\nu_k} (1 - x_1 - \ldots - x_k)^{m - \nu_1 - \ldots - \nu_k},$$

with

$$\binom{m}{\nu_1, \ldots, \nu_k} = \frac{m!}{\nu_1! \ldots \nu_k! (m - \nu_1 - \ldots - \nu_k)!},$$

where f and $p_{\nu_1, \ldots, \nu_k; m}$ are defined on k-dimensional simplex

$$\Delta_k : x_i \geq 0, i \in \{1, 2, \ldots, k\}, x_1 + \ldots + x_k \leq 1.$$

For $x \in \mathbb{R}$, $k \in \mathbb{N}$, let $x^{[k]} = x(x-1)\ldots(x-k+1)$, $x^{[0]} = 1$. It is well known that (see [4])

$$x^k = \sum_{\nu=1}^{k} S(k, \nu) x^{[\nu]}, \quad x \in \mathbb{R},\ k \in \mathbb{N}^*, \tag{1.2}$$

and

$$x^{[k]} = \sum_{\nu=1}^{k} s(k, \nu) x^{\nu}, \quad x \in \mathbb{R},\ k \in \mathbb{N}^*, \tag{1.3}$$

where $S(k, \nu)$, $\nu \in \{1, 2, \ldots, k\}$ are the Stirling numbers of second kind, and $s(k, \nu)$, $\nu \in \{1, 2, \ldots, k\}$ are the Stirling numbers of first kind. These numbers verify the relations (see[4])

$$S(p,k) = kS(p-1,k) + S(p-1,k-1),\ S(1,1) = 1, \tag{1.4}$$
$$S(2,1) = S(2,2) = 1,\ S(p,1) = S(p,p) = 1,$$

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for $p \in \mathbb{N}$, $p \geq 3$, $k \in \{2, 3, \ldots, p-1\}$, and

$$s(p,k) = s(p-1,k-1) - (p-1)s(p-1,k),\ s(1,1) = 1, \qquad (1.5)$$
$$s(2,1) = -1, s(2,2) = 1,$$

for $p \geq 3$, $k \in \{2, 3, \ldots, p-1\}$. We note $S(p,k) = 0$ and $s(p,k) = 0$, from definition, if $p, k \in \mathbb{N}, p < k$, or if $k = 0$ and $s(0,0) = 1$. In the paper [1] we proved that

Proposition 1.1. If $m, p_1, \ldots, p_k \in \mathbb{N}$, then

$$(B_m e_{p_1 \ldots p_k})(x_1, \ldots, x_k) = \frac{1}{m^{p_1 + \ldots + p_k}} \cdot \qquad (1.6)$$
$$\cdot \sum_{n_1=1}^{p_1} \ldots \sum_{n_k=1}^{p_k} m^{[n_1 + \ldots + n_k]} S(p_1, n_1) \ldots S(p_k, n_k) x_1^{n_1} \ldots x_k^{n_k},$$

where $e_{p_1 \ldots p_k}(x_1, \ldots, x_k) = x_1^{p_1} \ldots x_k^{p_k}$, $(x_1, \ldots, x_k) \in \Delta_k$.

2. Main results

Theorem 2.1. We have the formula

$$\sum_{\substack{\nu_1, \ldots, \nu_k \geq 0 \\ \nu_1 + \ldots + \nu_k \leq m}} \left(\frac{\nu_1}{m} - x_1\right) \ldots \left(\frac{\nu_k}{m} - x_k\right) p_{\nu_1, \ldots, \nu_k; m}(x_1, \ldots, x_k) = \frac{x_1 \ldots x_k}{m^k} \sum_{i=0}^{k} a_i m^i \qquad (2.7)$$

where

$$a_i = \sum_{j=0}^{i} (-1)^j \binom{k}{j} s(k-j, i-j), \qquad (2.8)$$

$i \in \{0, 1, \ldots, k\}$.

Proof. Because

$$\left(\frac{\nu_1}{m} - x_1\right) \ldots \left(\frac{\nu_k}{m} - x_k\right) = \frac{\nu_1}{m} \ldots \frac{\nu_k}{m} - \sum_{i=1}^{k} \frac{\nu_1}{m} \ldots x_i \ldots \frac{\nu_k}{m} +$$
$$+ \sum_{\substack{i,j=1 \\ i<j}}^{k} \frac{\nu_1}{m} \ldots x_i \ldots x_j \ldots \frac{\nu_k}{m} - \ldots + (-1)^k x_1 \ldots x_k,$$

we have

$$\sum_{\substack{\nu_1,\ldots,\nu_k\geq 0\\ \nu_1+\ldots+\nu_k\leq m}} \left(\frac{\nu_1}{m}-x_1\right)\ldots\left(\frac{\nu_k}{m}-x_k\right)p_{\nu_1,\ldots,\nu_k;m}(x_1,\ldots,x_k) =$$

$$= (B_m e_{11\ldots 1})(x_1,\ldots x_k) - \sum_{i=1}^{k} x_i (B_m e_{1\ldots 0\ldots 1})(x_1,\ldots,x_k) + \ldots +$$

$$+ (-1)^k x_1\ldots x_k (B_m e_{00\ldots 0})(x_1,\ldots,x_k) =$$

$$= \left(\binom{k}{0}\frac{m^{[k]}}{m^k} - \binom{k}{1}\frac{m^{[k-1]}}{m^{k-1}} + \ldots + (-1)^k\binom{k}{k}\right) x_1\ldots x_k =$$

$$= \frac{x_1x_2\ldots x_k}{m^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}m^i m^{[k-i]} = \frac{x_1x_2\ldots x_k}{m^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}m^i\cdot$$

$$\cdot\sum_{j=1}^{k-i}s(k-i,j)m^j = \frac{x_1x_2\ldots x_k}{m^k}\sum_{i=0}^{k}\left(\sum_{j=0}^{i}(-1)^j\binom{k}{j}s(k-j,i-j)\right)m^i.$$

We used Proposition 1.1, from where we have

$$(B_m e_{11\ldots 1})(x_1,\ldots,x_k) = \frac{m^{[k]}}{m^k}x_1\ldots x_k; x_i(B_m e_{1\ldots 0\ldots 1})(x_1,\ldots,x_k) =$$

$$= \frac{m^{[k-1]}}{m^{k-1}}x_1\ldots x_k, i\in\{1,2,\ldots,k\};$$

$$x_ix_j(B_m e_{1\ldots 0\ldots 0\ldots 1})(x_1,\ldots,x_k) = \frac{m^{[k-2]}}{m^{k-2}}x_1\ldots x_i\ldots x_j\ldots x_k$$

$$i,j\in\{1,2,\ldots,k\}, i<j, ..., x_1\ldots x_k(B_m e_{0\ldots 0})(x_1,\ldots,x_k) = x_1\ldots x_k.$$

□

Remark 2.1. Obviously, $a_0 = 0$. From (1.5) we have that $s(k,1) = (-1)^{k-1}(k-1)!$ so that $a_1 = (-1)^{k-1}(k-1)!$, for $k\geq 2$; for $k=1$ we have $a_1 = s(1,1) - s(0,0) = 0$.

Remark 2.2. Because $s(k,k) = 1$ we have

$$a_k = \sum_{j=0}^{k}(-1)^j\binom{k}{j} = 0.$$

Remark 2.3. From (1.5) we can easily obtain that $s(k,k-1) = -\frac{k(k-1)}{2}$, $k\geq 1$. So that we have for $k\geq 3$

$$a_{k-1} = \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}s(k-j,k-j-1) = -\sum_{j=0}^{k-1}(-1)^j\frac{k!}{j!(k-j)!}\cdot$$

$$\cdot\frac{(k-j)(k-j-1)}{2} = -\frac{k(k-1)}{2}\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j} = 0.$$

Particular cases:

1) For $k=1$ we have

$$\sum_{i=0}^{m}\left(\frac{i}{m}-x\right)p_{i;m}(x)=\frac{a_0+a_1m}{m}x=0. \tag{2.9}$$

2) For $k=2$ we obtain the formula

$$\sum_{i=0}^{m}\sum_{j=0}^{m-i}\left(\frac{i}{m}-x\right)\left(\frac{j}{m}-y\right)p_{i,j;m}(x,y)=\frac{a_0+a_1m+a_2m^2}{m^2}xy=-\frac{xy}{m}, \tag{2.10}$$

demonstrated in [3]

3) For $k=3$ we obtain

$$\sum_{\substack{i,j,k\geq 0\\ i+j+k\leq m}}^{m}\left(\frac{i}{m}-x\right)\left(\frac{j}{m}-y\right)\left(\frac{k}{m}-z\right)p_{i,j,k;m}(x,y,z)= \tag{2.11}$$

$$=\frac{a_0+a_1m+a_2m^2+a_3m^3}{m^3}xyz=\frac{2xyz}{m^2}$$

because $a_1=(-1)^2\cdot 2!=2$ and $a_2=0$ from Remark 2.3.

4) For $k=4$ we obtain

$$\sum_{\substack{\nu_1,\nu_2,\nu_3,\nu_4\geq 0\\ \nu_1+\nu_2+\nu_3+\nu_4\leq m}}^{m}\left(\frac{\nu_1}{m}-x_1\right)\left(\frac{\nu_2}{m}-_2\right)\left(\frac{\nu_3}{m}-x_3\right)\left(\frac{\nu_4}{m}\right)p_{\nu_1,\nu_2,\nu_3,\nu_4;m}(x_1,x_2,x_3,x_4)= \tag{2.12}$$

$$=\frac{a_0+a_1m+a_2m^2+a_3m^3+a_4m^4}{m^4}x_1x_2x_3x_4=\frac{3m-6}{m^3}x_1x_2x_3x_4$$

because

$$a_1=(-1)^3\cdot 3!=-6a_2=\binom{4}{0}\cdot s(4,2)-\binom{4}{1}\cdot s(3,1)+\binom{4}{2}\cdot s(2,0)=3$$

and $a_3=0$ from Remark 2.3.

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