CREATIVE MATH. & INF.
17 (2008), 25 - 32

Anticommutativity in the ring of square matrices of the second order with complex entries

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ABSTRACT. The main purpose of this paper is to solve the equation $AB+BA=0$ in the ring $M_2(\mathbb{C})$ of square matrices of the second order with complex entries. The discussion is made by considering the cases when A and B are inversable or singular. The methods used in each case are completely different and instructive. Considerations about matrices which commutes and finally an application are also given.

1. INTRODUCTION

We say that a pair (A,B) of two matrices $A,B\in M_2(\mathbb{C})$ is an anticommutative pair if $AB=-BA$. Obviously, the problem of finding anticommutative pairs (A,B) is equivalent with the problem of solving the equation

$$AB+BA=0_2.$$

We denote the zero matrix, respective the unity matrix by

$$0_2=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad I_2=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For a matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in M_2(\mathbb{C})$ we denote by

$$\text{det}A=ad-bc \quad , \quad \text{tr}A=a+d$$

the determinant of A, respective the trace of A. We also define

$$A^*=\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

the adjoint of the matrix A, which satisfies

$$A\cdot A^*=A^*\cdot A=(\det A)\cdot I_2.$$

For every matrices $X,Y\in M_2(\mathbb{C})$ it holds

$$\text{tr}(XY)=\text{tr}(YX).$$

For each matrix $A\in M_2(\mathbb{C})$ it can be easily established the relation

$$A^2-\text{tr}A\cdot A+\ \text{det}A\cdot I_2=0_2$$

also called Hamilton-Cayley relation.

Received: 29.03.2007. In revised form: 2.11.2007.
2000 *Mathematics Subject Classification.* 15A27, 15A99.
Key words and phrases. *Matrices, determinants, Hamilton-Cayley relation, commutativity, anticommutativity.*

As a direct consequence, to each matrix $A \in M_2(\mathbb{C})$ we can inductively assign two sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ of complex numbers for which

$$A^n = a_n A + b_n I_2 \quad , \quad \forall\, n \geq 1.$$

Case $n = 1$ is trivial while case $n = 2$ follows from Hamilton-Cayley relation (e. g. [1], [2]).

Next we give the following

Lemma 1.1. *Let $A, B \in M_2(\mathbb{C})$ be two matrices, not of the form λI_2, with $\lambda \in \mathbb{C}$. If $AB = BA$, then*

$$B = \alpha A + \beta I_2,$$

for some complex numbers α, β, $\alpha \neq 0$.

Proof. If denote $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, then from $AB = BA$ we derive

$$\frac{y}{b} = \frac{z}{c} = \frac{x-t}{a-d} = \alpha$$

(with the convention that if a denominator is zero, then the corresponding denumerator is also zero). At least one of the fraction is not $\dfrac{0}{0}$, because $b, c, a-d$ cannot be all zero. Thus α is well defined. It follows

$$y = \alpha b \quad , \quad z = \alpha c \quad , \quad t = \alpha d + \beta \quad , \quad x = \alpha a + \beta,$$

with $\beta = t - \alpha d$. Finally,

$$B = \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} \alpha a + \beta & \alpha b \\ \alpha c & \alpha d + \beta \end{pmatrix} = \alpha A + \beta I_2.$$

□

We also give

Lemma 1.2. *The solutions of the equation*

$$X^2 = -I_2 \quad , \quad X \in M_2(\mathbb{C}) \tag{1.1}$$

are $X = iI_2$, $X = -iI_2$ and any matrix of the form

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

with $a^2 + bc = -1$.

Proof. With the notation $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the given equation can be reduced to the following system

$$\begin{cases} a^2 + bc & = & -1 \\ b(a+d) & = & 0 \\ c(a+d) & = & 0 \\ d^2 + bc & = & -1 \end{cases}$$

If $a+d \neq 0$, then $b=c=0$. Thus $X = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, with $a^2 = d^2 = -1$. If $a+d=0$, then $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, with $a^2 + bc = -1$. $\square$

2. THE RESULTS

If two matrices $A, B \in M_2(\mathbb{C})$ commutes, $AB = BA$ then

$$A^m B^n = B^n A^m$$

for all positive integers m, n. Moreover, the reciprocal part is true in the sense of the following

Lemma 2.1. *Let $A, B \in M_2(\mathbb{C})$ be two matrices. Assume that for some positive integers m, n we have $A^m B^n = B^n A^m$ and the matrices A^m and B^n are not of the form λI_2, $\lambda \in \mathbb{C}$. Then $AB = BA$.*

Proof. As we stated, we can define sequences $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}, (c_n)_{n\geq 1}, (d_n)_{n\geq 1}$ of complex numbers such that

$$A^k = a_k A + b_k I_2 \quad , \quad B^k = c_k B + d_k I_2,$$

for all positive integers k. From the hypothesis, $a_m \neq 0$ and $c_n \neq 0$. Then

$$A^m B^n = B^n A^m \Rightarrow$$
$$\Rightarrow (a_m A + b_m I_2)(c_n B + d_n I_2) = (c_n B + d_n I_2)(a_m A + b_m I_2) \Rightarrow$$
$$\Rightarrow a_m c_n (AB - BA) = 0_2.$$

Hence $AB = BA$, because $a_m c_n \neq 0$.

With these preparations, we can solve the equation

$$AB + BA = 0_2. \tag{2.1}$$

$\square$

First we assume that both matrices A and B are inversable.

Theorem 2.1. *The solutions of the equation $AB + BA = 0_2$, with $A, B \in M_2(\mathbb{C})$ inversable, are*

$$A = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \quad , \quad B = A^{-1} \cdot \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

where x, y, z, a, b, c are complex numbers with $x^2 + yz \neq 0$, $a^2 + bc \neq 0$ and $2ax + bz + cy = 0$.

Proof. We can assume without loss of generality that

$$\det A = \det B = 1.$$

Indeed, this can be made by replacing A with $\frac{1}{\det A} \cdot A$ and B with $\frac{1}{\det B} \cdot B$. We have

$$\mathrm{tr}(AB + BA) = 0 \Rightarrow \mathrm{tr}\,(AB) + r\,(BA) = 0$$

and from $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$, we obtain

$$\mathrm{tr}(AB) = 0.$$

We also have

$$\det(AB) = \det A \cdot B = 1$$

and from Hamilton-Cayley relation we deduce

$$(AB)^2 = -I_2. \tag{2.2}$$

Now we can use Lemma 1.2 to solve the equation (2.1). If X denotes any solution of the equation (1.1), then

$$AB = X \Rightarrow B = A^{-1}X.$$

With $B = A^{-1}X$, the equation (2.1) becomes

$$A \cdot A^{-1}X + A^{-1}X \cdot A = 0_2.$$

By multiplying with A to the left, we derive

$$AX + XA = 0_2. \tag{2.3}$$

Cases $X = iI_2$ and $X = -iI_2$ are not acceptable, because A is inversable. In consequence, if

$$A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \quad , \quad X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

with $x, y, z, t, a, b, c \in \mathbb{C}$, $a^2 + bc = -1$, then the condition (2.3) becomes

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} = 0_2.$$

In terms of linear systems, we obtain

$$\begin{cases} 2ax + bz + cy & = 0 \\ b(t+x) & = 0 \\ c(t+x) & = 0 \\ -2at + bz + cy & = 0 \end{cases} .$$

If $t + x \neq 0$, then $b = c = 0$. From the first and the last equation of the system we deduce $a = 0$. This is impossible, because $a^2 + bc = -1$.

In consequence, $t + x = 0$. In this case, the first and the last equation of the system are equivalent. It follows

$$A = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \quad , \quad X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

for any $x, y, z, t, a, b, c \in \mathbb{C}$, satisfying

$$a^2 + bc = -1 \quad , \quad 2ax + bz + cy = 0.$$

The general solution of the given equation is $(\zeta A, \mu B)$, where $\zeta, \mu \in \mathbb{C}^*$. □

Theorem 2.2. *The solutions of the equation* $AB + BA = 0_2$, $A, B \in M_2(\mathbb{C})$, *with* A *inversable,* B *singular,* $B \neq 0_2$ *are*

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \quad , \quad B = \begin{pmatrix} x & y \\ z & -x \end{pmatrix},$$

where a, b, c, x, y, z *are complex numbers with* $x^2 + yz = 0$, $a^2 + bc \neq 0$ *and* $2ax + bz + cy = 0$.

Proof. By multiplying the given equation with B to the right, then with B to the left, we obtain

$$(AB + BA)B = 0_2 \Rightarrow AB^2 + BAB = 0_2$$
$$B(AB + BA) = 0_2 \Rightarrow BAB + B^2A = 0_2.$$

Hence

$$AB^2 = B^2A. \tag{2.4}$$

If A and B^2 are not of the form λI_2, $\lambda \in \mathbf{C}$, then $AB = BA$, according to the Lemma 2.1. But $AB + BA = 0_2$, so

$$AB = BA = 0_2.$$

By multiplying with A^{-1}, we obtain $B = 0_2$.

If $A = \lambda I_2$, $\lambda \neq 0$, then easy $B = 0_2$.

If $B^2 = \lambda I_2$, then $\lambda = 0$, because B is assumed to be singular. From $B^2 = 0_2$ and $B \neq 0_2$ it results tr$B = 0$. If denote

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad B = \begin{pmatrix} x & y \\ z & -x \end{pmatrix},$$

with $a, b, c, d, x, y, z \in \mathbb{C}$, $x^2 + yz = 0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & -x \end{pmatrix} + \begin{pmatrix} x & y \\ z & -x \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0_2.$$

Therefore

$$\begin{cases} 2ax + cy + bz & = 0 \\ y(a+d) & = 0 \\ z(a+d) & = 0 \\ -2dx + cy + bz & = 0 \end{cases}.$$

If $a + d \neq 0$, then $y = z = 0$ and $x = 0$. It follows $B = 0_2$.

If $a + d = 0$, then the previous system becomes equivalent with

$$2ax + cy + bz = 0.$$

□

Finally, we consider the case when A, B are both singular.

Theorem 2.3. *Let $A, B \in M_2(\mathbb{C})$ be singular such that $AB + BA = 0_2$. Then $A = 0_2$ or $B = 0_2$ or*

$$B = \lambda A^*$$

for some complex number λ.

Proof. We assume that $A \neq 0_2$ and $B \neq 0_2$. Let us denote

$$A = \begin{pmatrix} xy & xz \\ yt & zt \end{pmatrix} \quad , \quad B = \begin{pmatrix} mn & mp \\ nq & pq \end{pmatrix},$$

where x, y, z, t, m, n, p, q are complex numbers. Then

$$\begin{pmatrix} xy & xz \\ yt & zt \end{pmatrix}\begin{pmatrix} mn & mp \\ nq & pq \end{pmatrix} + \begin{pmatrix} mn & mp \\ nq & pq \end{pmatrix}\begin{pmatrix} xy & xz \\ yt & zt \end{pmatrix} = 0_2$$

is equivalent with the system

$$\begin{cases} my(nx+pt)+nx(my+qz) & =0 \\ mp(xy+zt)+xz(mn+pq) & =0 \\ yt(mn+pq)+nq(xy+zt) & =0 \\ qz(nx+pt)+pt(my+qz) & =0 \end{cases}.$$

By adding the first and the last equation of the system, we obtain

$$(my+qz)(nx+pt)=0$$

and we will prove that $my+qz=0$ and $nx+pt=0$.

In this sense, let us suppose by contrary that $my+qz=0$ and $nx+pt\neq 0$. From the last equation of the system, we derive $qz=0$ and so $my=0$.

If $y=z=0$, then $A=0_2$ and if $m=q=0$, then $B=0_2$.

If for example $q=y=0$, then from the second equation of the system, we obtain

$$mz(nx+pt)=0 \Rightarrow mz=0.$$

If $m=0$, then $B=0_2$ and if $z=0$, then $A=0_2$.

Now we are in case

$$my+qz=0 \quad , \quad nx+pt=0. \tag{2.5}$$

This conditions are sufficient for variables x,y,z,t,m,n,p,q to satisfy the system.

If $x=t=0$, then $A=0_2$ so we can assume that $x\neq 0$. Similarly, if $y=z=0$, then $A=0_2$, so we can assume that $y\neq 0$. Under these assumptions,

$$m=-\frac{qz}{y} \quad , \quad n=-\frac{pt}{x},$$

accordingly to (2.5). Finally, with these values for m and n, we derive

$$B=\frac{pq}{xy}\cdot\begin{pmatrix} zt & -xz \\ -ty & xy \end{pmatrix}=\frac{pq}{xy}\cdot A^*.$$

□

3. Applications

The above theoretical results can be successfully used to establish other theoretical results or in practical problems. To show this, we will consider here the problem of solving the equation

$$AY+YA=f \quad , \quad Y\in M_2(\mathbb{C}), \tag{3.1}$$

where $A\in M_2(\mathbb{C})$ and $f\in M_2(\mathbb{C})$ are given. In the previous section of our work, we have solved this kind of equations in case $f=0_2$. As in theory of differential equations, the equation (3.1) is close related with the corresponding homogeneous equation

$$AX+XA=0_2. \tag{3.2}$$

Indeed, we assert that the problem of solving equation (3.1) can be reduced to an easier problem of finding a particular solution, as we can see from the following

Theorem 3.1. *Let there be given two matrices $A, f \in M_2(\mathbb{C})$. Assume that $Y_0 \in M_2(\mathbb{C})$ is a solution the equation (3.1). Then every solution $Y \in M_2(\mathbb{C})$ of the equation (3.1) can be written as*

$$Y = Y_0 + X,$$

where $X \in M_2(\mathbb{C})$ is a solution of the homogeneous equation (3.2).

Proof. First,

$$AY + YA = A(Y_0 + X) + (Y_0 + X)A =$$

$$= (AY_0 + Y_0A) + (AX + XA) = f + 0_2 = f.$$

Reciprocally, from the fact that Y_0 is solution of (3.1), we deduce

$$AY_0 + Y_0A = f.$$

If Y is another solution, then

$$AY + YA = f$$

and by substraction, we obtain

$$A(Y - Y_0) + (Y - Y_0)A = 0_2.$$

If denote $X = Y - Y_0$ then $Y = Y_0 + X$ and X is solution of (3.2). □

Further in case $f = I_2$, we completely solve the equation

$$AY + YA = I_2. \tag{3.3}$$

Also here we need a discussion relative to $\det A$. First, if A is inversable, then a particular solution of (3.3) is

$$Y_0 = \frac{1}{2} \cdot A^{-1}.$$

From Theorems 2.1-2.2 it results the following

Theorem 3.2. *Let there be given a matrix $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in M_2(\mathbb{C})$, with $a^2 + bc \neq 0$ and let $Y \in M_2(\mathbb{C})$ be a solution of the equation (3.3). Then*

$$Y = A^{-1} \cdot \begin{pmatrix} \frac{1}{2} + x & y \\ z & \frac{1}{2} - x \end{pmatrix},$$

for some complex numbers x, y, z satisfying $x^2 + yz \neq 0$ and $2ax + bz + cy = 0$, or

$$Y = \begin{pmatrix} \frac{1}{2} + u & v \\ w & \frac{1}{2} - u \end{pmatrix},$$

for some complex numbers u, v, w satisfying $u^2 + vw = 0$ and $2au + bw + cv = 0$.

The case when A is singular follows from Theorem 2.3.

Let $A=\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\in M_2(\mathbb{C})$, $A\neq 0_2$, be with $a^2+bc=0$. It follows $b\neq 0$ or $c\neq 0$, because $A\neq 0_2$. Now it can be easily verified that a particular solution of (3.3) is

$$Y_0=\begin{pmatrix} 0 & 0 \\ \frac{1}{b} & 0 \end{pmatrix} \quad , \quad if \;\; b\neq 0$$

and

$$Y_0=\begin{pmatrix} 0 & \frac{1}{c} \\ 0 & 0 \end{pmatrix} \quad , \quad if \;\; c\neq 0.$$

We can state the following

Theorem 3.3. *Let there be given* $A=\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\in M_2(\mathbb{C})$, $A\neq 0_2$, *with* $a^2+bc=0$ *and assume without loss of generality that* $b\neq 0$. *Let* $Y\in M_2(\mathbb{C})$ *be a solution of the equation (3.3). Then*

$$Y=\begin{pmatrix} 0 & 0 \\ \frac{1}{b} & 0 \end{pmatrix}+\lambda A^*,$$

for some complex number λ.

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