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Some stability and strong convergence results for the Jungck-Ishikawa iteration process

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ABSTRACT. In this paper, we shall establish some stability results as well as some strong convergence results for a pair of nonselfmappings using a newly introduced Jungck-Ishikawa iteration process and some general contractive conditions. Our results are generalizations and extensions of the results in some of the references listed in the reference section of this paper as well as of some other analogous ones in the literature.

1. INTRODUCTION

Let (E, d) be a complete metric space and $T : E \to E$ a selfmap of E. Suppose that $F_T = \{ p \in E \mid Tp = p \}$ is the set of fixed points of T.
There are several iteration processes in the literature for which the fixed points of operators have been approximated over the years by various authors. In a complete metric space, the Picard iteration process $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n,\ n = 0, 1, \cdots, \tag{1.1}$$

has been employed to approximate the fixed points of mappings satisfying the inequality relation

$$d(Tx, Ty) \leq \alpha d(x, y),\ \forall\, x,\ y \in E \text{ and } \alpha \in [0, 1). \tag{1.2}$$

Condition (1.2) is called the *Banach's contraction condition.* Any operator satisfying (1.2) is called *strict contraction.* Also, condition (1.2) is significant in the celebrated Banach's fixed point Theorem [2].

In the Banach space setting, we shall state some of the iteration processes generalizing (1.1) as follows: For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n,\ n = 0, 1, \cdots, \tag{1.3}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1]$, is called the Mann iteration process (see Mann [19]).
For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\left.\begin{array}{l} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tz_n \\ z_n = (1 - \beta_n)x_n + \beta_n Tx_n \end{array}\right\} n = 0, 1, \cdots, \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$, is called the Ishikawa iteration process (see Ishikawa [11]).
The following is the iteration process introduced by Singh et al [38] to establish some stability results: Let S and T be operators on an arbitrary set Y with values

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in E such that $T(Y) \subseteq S(Y)$. $S(Y)$ is a complete subspace of E. Then, for $x_0 \in Y$, the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tx_n,\ n = 0, 1, \cdots, \tag{1.5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ is called the *Jungck-Mann* iteration process.
If $\alpha_n = 1$ and Y = E in (1.5), then we obtain

$$Sx_{n+1} = Tx_n,\ n = 0, 1, 2, \cdots, \tag{1.6}$$

which is the *Jungck iteration.* See Jungck [13] for detail.
While the iteration process (1.5) extends (1.1), (1.3) and (1.6), the iteration processes (1.4) and (1.5) are independent.

Kannan [14] established an extension of the Banach's fixed point theorem by using the following contractive definition: For a selfmap T, there exists $\beta \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \beta\left[d(x, Tx) + d(y, Ty)\right],\ \forall\ x,\ y \in E. \tag{1.7}$$

Chatterjea [6] used the following contractive condition: For a selfmap T, there exists $\gamma \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \gamma\left[d(x, Ty) + d(y, Tx)\right],\ \forall\ x,\ y \in E. \tag{1.8}$$

Zamfirescu [39] established a nice generalization of the Banach's fixed point theorem by combining (1.2), (1.7) and (1.8). That is, for a mapping $T : E \to E$, there exist real numbers $\alpha,\ \beta, \gamma$ satisfying $0 \leq \alpha < 1,\ 0 \leq \beta < \frac{1}{2},\ 0 \leq \gamma < \frac{1}{2}$ respectively such that for each $x,\ y \in E$, at least one of the following is true:

$$\left.\begin{array}{l} (z_1)\ d(Tx, Ty) \leq \alpha d(x, y) \\ (z_2)\ d(Tx, Ty) \leq \beta\left[d(x, Tx) + d(y, Ty)\right] \\ (z_3)\ d(Tx, Ty) \leq \gamma\left[d(x, Ty) + d(y, Tx)\right]. \end{array}\right\} \tag{1.9}$$

The mapping $T : E \to E$ satisfying (1.9) is called the *Zamfirescu contraction.* Any mapping satisfying condition (z_2) of (1.9) is called a *Kannan mapping,* while the mapping satisfying condition (z_3) is called *Chatterjea operator.* The contractive condition (1.9) implies

$$||Tx - Ty|| \leq 2\delta||x - Tx|| + \delta||x - y||,\ \forall\ x,\ y \in E, \tag{1.10}$$

where $\delta = \max\left\{\alpha, \dfrac{\beta}{1-\beta}, \dfrac{\gamma}{1-\gamma}\right\},\ 0 \leq \delta < 1.$
Condition (1.9) was used by Rhoades [33, 34] to obtain some convergence results for Mann and Ishikawa iteration processes in a uniformly convex Banach space. The results of [33, 34] were recently extended by Berinde [5] to an arbitrary Banach space for the same fixed point iteration processes. Similar convergence results were also established in Rafiq [28, 29] for other interesting iteration processes.

Singh et al [38] defined the following general iteration process:
Let $S,\ T : Y \to E$ and $T(Y) \subseteq S(Y)$. For any $x_0 \in Y$, let

$$Sx_{n+1} = f(T, x_n),\ n = 0, 1, \cdots \tag{1.11}$$

where f is some function.

For $f(T, x_n) = Tx_{n+1}$, then (1.11) reduces to the Jungck-type iteration process of Singh et al [38].

If $Y = E$, and $f(T, x_n) = Tx_n,\ n = 0, 1, \cdots$, then (1.11) reduces to the Jungck iteration process of (1.6).
If $Y = E$, and $f(T, x_n) = Tx_n,\ n = 0, 1, \cdots$, then (1.11) reduces to the Jungck iteration process of (1.6). Jungck [13] established that the maps S and T satisfying

$$d(Tx, Ty) \leq ad(Sx, Sy),\ \forall\, x,\ y \in E,\ a \in [0, 1), \tag{1.12}$$

have a unique common fixed point in a complete metric space E, provided that S and T commute, $T(Y) \subseteq S(Y)$ and S is continuous. For results which are similar to Jungck [13] in uniform spaces, we refer to Aamri and El Moutawakil [1] as well as to Olatinwo [23, 24].

The following definition of the stability of iteration process due to Singh et al [38] shall be required in the sequel.

Definition 1.1. Let $S,\ T : Y \to E,\ T(Y) \subseteq S(Y)$ and z a coincidence point of S and T, that is, $Sz = Tz = p$ (say). For any $x_0 \in Y$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$, generated by the iteration procedure (1.11) converge to p. Let $\{Sy_n\}_{n=0}^{\infty} \subset E$ be an arbitrary sequence, and set $\epsilon_n = d(Sy_{n+1}, f(T, y_n)),\ n = 0, 1, \cdots$ Then, the iteration procedure (1.11) will be called $(S, T)-$stable if and only if $\lim\limits_{n\to\infty} \epsilon_n = 0$ implies that $\lim\limits_{n\to\infty} Sy_n = p$.

This definition reduces to that of the stability of iteration procedure due to Harder and Hicks [9] when $Y = E$ and $S = I$ (identity operator).

Several stability results established in metric space and normed linear space are available in the literature. Some of the various authors whose contributions are of important value in the study of stability of the fixed point iteration procedures are Ostrowski [27], Harder and Hicks [9], Rhoades [30, 32], Osilike [25], Osilike and Udomene [26], Jachymski [12], Berinde [3, 4] and Singh et al [38]. Harder and Hicks [9], Rhoades [30, 32], Osilike [25] and Singh et al [38] used the method of the summability theory of infinite matrices to prove various stability results for certain contractive definitions. The method has also been adopted to establish various stability results for certain contractive definitions in Olatinwo et al [20, 21]. Osilike and Udomene [26] introduced a shorter method of proof of stability results and this has also been employed by Berinde [3], Imoru and Olatinwo [10], Olatinwo et al [22] and some others. In Harder and Hicks [9], the contractive definition stated in (1.2) was used to prove a stability result for the Kirk's iteration process. The first stability result on $T-$ stable mappings was proved by Ostrowski [27] for the Picard iteration using condition (1.2).

In addition to (1.2), the contractive condition in (1.9) was also employed by Harder and Hicks [9] to establish some stability results for both Picard and Mann iteration processes. Rhoades [30, 32] extended the stability results of [9] to more general classes of contractive mappings. Rhoades [30] extended the results of [9] to the following independent contractive condition: there exists $c \in [0, 1)$ such that

$$d(Tx, Ty) \leq c \max\left\{d(x, y), d(x, Ty), d(y, Tx)\right\},\ \forall\, x,\ y \in E. \tag{1.13}$$

Rhoades [32] used the following contractive definition: there exists $c \in [0, 1)$ such that $\forall\, x, y \in E$, we have

$$d(Tx,Ty) \le c\max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, d(x,Ty), d(y,Tx)\right\}. \qquad (1.14)$$

Moreover, Osilike [25] generalized and extended some of the results of Rhoades [32] by using a more general contractive definition than those of Rhoades [30, 32]. Indeed, he employed the following contractive definition: there exist $a \in [0,1],\ L \ge 0$ such that

$$d(Tx,Ty) \le Ld(x,Tx) + ad(x,y),\ \forall\ x,\ y \in E. \qquad (1.15)$$

Osilike and Udomene [26] introduced a shorter method to prove stability results for the various iteration processes using the condition (1.15). Berinde [3] established the same stability results for the same iteration processes using the same set of contractive definitions as in Harder and Hicks [9] but the same method of shorter proof as in Osilike and Udomene [26].

More recently, Imoru and Olatinwo [10] established some stability results which are generalizations of some of the results of [3, 9, 25, 26, 30, 32]. In Imoru and Olatinwo [10], the following contractive definition was employed: there exist $a \in [0,1)$ and a monotone increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$, such that

$$d(Tx,Ty) \le \varphi(d(x,Tx)) + ad(x,y),\ \forall\ x,\ y \in E. \qquad (1.16)$$

Condition (1.16) was also employed in Olatinwo et al [20] to establish some stability results in normed linear space setting with additional condition of continuity imposed on φ.

However, Singh et al [38] established some stability results for Jungck and Jungck-Mann iteration processes by employing two contractive definitions both of which generalize those of Osilike [25] but is independent of that of Imoru and Olatinwo [10]. Singh et al [38] obtained stability results for Jungck and Jungck-Mann iterative procedures in metric space using both the contractive definition (1.12) and the following: For $S,\ T : Y \to E$ and some $a \in [0,1)$, we have

$$d(Tx,Ty) \le ad(Sx,Sy) + Ld(Sx,Tx),\ \forall\ x,\ y \in Y. \qquad (1.17)$$

In the next section, we shall introduce the Jungck-Ishikawa iteration process to prove some stability and convergence results for nonselfmappings in normed linear space and arbitrary Banach space respectively. In establishing our results, a more general contractive condition than (1.9) will be considered.

2. Preliminaries

We shall consider the following iteration process in establishing our results: Let $(E, ||.||)$ be a Banach space and Y an arbitrary set. Let $S,\ T : Y \to E$ be two nonselfmappings such that $T(Y) \subseteq S(Y),\ S(Y)$ is a complete subspace of E and S is injective. Then, for $x_0 \in Y$, define the sequence $\{Sx_n\}_{n=0}^{\infty}$ iteratively by

$$\left.\begin{array}{l} Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tz_n \\ Sz_n = (1-\beta_n)Sx_n + \beta_n Tx_n \end{array}\right\},\ n = 0, 1, \cdots, \qquad (2.1)$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.
The iteration process (2.1) will be called the *Jungck-Ishikawa* iteration process. The

iteration processes (1.1) and (1.3) - (1.6) are special cases of (2.1). For instance, if in (2.1), S is identity operator, $Y = E,\ \beta_n = 0$ then we obtain the Mann iteration process of (1.3). Since S is injective, if $\beta_n = 0$, then for $x_0 \in Y$, (2.1) reduces to the Jungck-Mann iteration process of (1.5).

In addition to the iteration process (2.1), we shall employ the following contractive definitions:

Definition 2.1. For two nonselfmappings $S,\ T : Y \to E$ with $T(Y) \subseteq S(Y)$, where $S(Y)$ is a complete subspace of E,
(a) there exist a real number $a \in [0, 1)$ and a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and $\forall$ x, y $\in$ Y, we have

$$||Tx - Ty|| \leq \varphi(||Sx - Tx||) + a||Sx - Sy||; \tag{2.2}$$

and
(b) there exist real numbers $M \geq 0,\ a \in [0, 1)$ and a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and $\forall$ x, y $\in$ Y, we have

$$||Tx - Ty|| \leq \frac{\varphi(||Sx - Tx||) + a||Sx - Sy||}{1 + M||Sx - Tx||}. \tag{2.3}$$

In this paper, we shall consider the Jungck-Ishikawa iteration process defined in (2.1) to establish some stability results for nonselfmappings in normed linear space as well as obtain some strong convergence results for these nonselfmappings in an arbitrary Banach space by employing the contractive conditions (2.2) and (2.3). Our stability results are generalizations and extensions of those of Singh et al [38], some results of [3, 10, 20, 21, 22, 30, 32], while the convergence results extend, generalize and improve those of [5, 15, 16, 33, 34]. For more on the study of fixed point iteration processes and various contractive conditions, our interested readers can consult Berinde [4], Ciric [7, 8], Rhoades [35] and others in the reference section of this paper.

Definition 2.2. Let X and Y be two nonempty sets and $S,\ T : X \to Y$ two mappings. Then, an element $x^* \in X$ is a *coincidence point* of S and T if and only if $Sx^* = Tx^*$. Denote the set of the coincidence points of S and T by $C(S,T)$.

There are several papers and monographs on the coincidence point theory. However, we refer our readers to Rus [36] and Rus et al [37] for the Definition 2.2 and some coincidence point results.

We shall require the following lemma in the sequel.

Lemma 2.1. (Berinde [3, 4]): *If δ is a real number such that $0 \leq \delta < 1$, and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n,\ n = 0, 1, \cdots,$$

we have $\lim_{n\to\infty} u_n = 0$.

We establish our main results in the next two sections. Section 3 deals with some stability results in normed linear space, while some strong convergence results are proved in Section 4.

3. Some stability results in normed linear space

Theorem 3.1. *Let $(E, ||.||)$ be a normed space and Y an arbitrary set. Suppose that S, $T : Y \to E$ are nonself operators such that $T(Y) \subseteq S(Y)$, $S(Y)$ a complete subspace of E,* and *S is an injective operator. Let z be a coincidence point of S and T (that is, $Sz = Tz = p$). Suppose that S and T satisfy condition (2.3). Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotone increasing function such that $\varphi(0) = 0$. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Ishikawa iteration process defined by (2.1) converging to p, where $\{\alpha_n\}_{n=0}^{\infty}$* and *$\{\beta_{\mathrm{n}}\}_{n=0}^{\infty}$ are sequences in $[0, 1]$ such that $0 < \alpha \leq \alpha_n$ and $0 < \beta \leq \beta_n$. Then, the Jungck-Ishikawa iteration process is (S, T)-stable.*

Proof. Suppose that $\{Sy_n\}_{n=0}^{\infty} \subset E$, $\epsilon_n = ||Sy_{n+1} - (1-\alpha_n)Sy_n - \alpha_n Tb_n||$, $n = 0, 1, \cdots$, where $Sb_n = (1-\beta_n)Sy_n + \beta_n Ty_n$ and let $\lim\limits_{n\to\infty} \epsilon_n = 0$. Then, we shall establish that $\lim\limits_{n\to\infty} Sy_n = p$, using the contractive condition (2.3) and the triangle inequality:

$$\begin{aligned} ||Sy_{n+1} - p|| &\leq ||Sy_{n+1} - (1-\alpha_n)Sy_n - \alpha_n Tb_n|| + ||(1-\alpha_n)Sy_n + \\ &\quad +\alpha_n Tb_n - (1-\alpha_n+\alpha_n)p|| \\ &= \epsilon_n + ||(1-\alpha_n)(Sy_n - p) + \alpha_n(Tb_n - p)|| \\ &\leq \epsilon_n + (1-\alpha_n)||Sy_n - p|| + \alpha_n||p - Tb_n|| \\ &= \epsilon_n + (1-\alpha_n)||Sy_n - p|| + \alpha_n||Tz - Tb_n|| \\ &\leq \epsilon_n + (1-\alpha_n)||Sy_n - p|| + \alpha_n[\frac{\varphi(||Sz - Tz||) + a||Sz - Sb_n||}{1 + M||Sz - Tz||}] \\ &= (1-\alpha_n)||Sy_n - p|| + a\alpha_n||p - Sb_n|| + \epsilon_n. \end{aligned} \tag{3.1}$$

Therefore, we have

$$\begin{aligned} ||p - Sb_n|| &= ||(1-\beta_n+\beta_n)p - (1-\beta_n)Sy_n - \beta_n Ty_n|| \\ &= ||(1-\beta_n)(p - Sy_n) + \beta_n(p - Ty_n)|| \\ &\leq (1-\beta_n)||p - Sy_n|| + \beta_n||Tz - Ty_n|| \\ &\leq (1-\beta_n + a\beta_n)||Sy_n - p||. \end{aligned} \tag{3.2}$$

Using (3.2) in (3.1) yields

$$\begin{aligned} ||Sy_{n+1} - p|| &\leq [1 - (1-a)\alpha_n - (1-a)a\alpha_n\beta_n]||Sy_n - p|| + \epsilon_n \\ &\leq [1 - (1-a)\alpha - (1-a)a\alpha\beta]||Sy_n - p|| + \epsilon_n. \end{aligned} \tag{3.3}$$

Since $0 \leq 1 - (1-a)\alpha - (1-a)a\alpha\beta < 1$, using Lemma 2.3 in (3.3) yields $\lim_{n\to\infty} ||Sy_n - p|| = 0$, that is, $\lim\limits_{n\to\infty} Sy_n = p$.
Conversely, let $\lim\limits_{n\to\infty} Sy_n = p$. Then, by using the triangle inequality and the contractive definition, we have the following:

$$\begin{aligned} \epsilon_n &= ||Sy_{n+1} - (1-\alpha_n)Sy_n - \alpha_n Tb_n|| \\ &\leq ||Sy_{n+1} - p|| + ||(1-\alpha_n+\alpha_n)p - (1-\alpha_n)Sy_n - \alpha_n Tb_n|| \\ &= ||Sy_{n+1} - p|| + ||(1-\alpha_n)(p - Sy_n) + \alpha_n(p - Tb_n)|| \\ &= ||Sy_{n+1} - p|| + ||(1-\alpha_n)(p - Sy_n) + \alpha_n(Tz - Tb_n)|| \\ &\leq ||Sy_{n+1} - p|| + (1-\alpha_n)||Sy_n - p|| + \alpha_n||Tz - Tb_n|| \\ &\leq ||Sy_{n+1} - p|| + (1-\alpha_n)||Sy_n - p|| + a\alpha_n||p - Sb_n||. \end{aligned} \tag{3.4}$$

Again, we have by the contractive condition that

$$\begin{aligned}||p-Sb_n|| &= ||(1-\beta_n+\beta_n)p-(1-\beta_n)Sy_n-\beta_nTy_n|| \\ &= ||(1-\beta_n)(p-Sy_n)+\beta_n(p-Ty_n)|| \\ &\leq (1-\beta_n)||p-Sy_n||+\beta_n||p-Ty_n|| \\ &= (1-\beta_n)||Sy_n-p||+\beta_n||Tz-Ty_n|| \\ &\leq (1-\beta_n+a\beta_n)||Sy_n-p||. \qquad (3.5)\end{aligned}$$

Using (3.5) in (3.4), then we obtain

$$\begin{aligned}\epsilon_n &\leq ||Sy_{n+1}-p||+[1-(1-a)\alpha_n-(1-a)a\alpha_n\beta_n]||Sy_n-p|| \\ &\leq ||Sy_{n+1}-p||+[1-(1-a)\alpha-(1-a)a\alpha\beta]||Sy_n-p|| \to 0 \text{ as n} \to \infty.\end{aligned}$$

Hence, the iteration process defined in (2.1) is stable with respect to the pair $(S,\, T)$. □

Theorem 3.2. *Let* $(E, ||.||)$ *be a normed space and* Y *an arbitrary set. Suppose that* $S,\ T : Y \to E$ *are nonself operators such that* $T(Y) \subseteq S(Y),\ S(Y)$ *a complete subspace of* E, and S *is an injective operator. Let* z *be a coincidence point of* S *and* T *(that is,* $Sz = Tz = p$*). Suppose that* S *and* T *satisfy condition (2.2). Let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a monotone increasing function such that* $\varphi(0) = 0$*. For* $x_0 \in Y$*, let* $\{Sx_n\}_{n=0}^{\infty}$ *be the Jungck-Ishikawa iteration process defined by (2.1) converging to* p*, where* $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_{\rm n}\}_{\rm n=0}^{\infty}$ *are sequences in* $[0, 1]$ *such that* $0 < \alpha \leq \alpha_n$ *and* $0 < \beta \leq \beta_n$*. Then, the Jungck-Ishikawa iteration process is* (S, T)*-stable.*

Proof. Theorem 3.2 is proved by putting $M = 0$ in the proof of Theorem 3.1. □

Remark 3.1. Both Theorem 3.1 and Theorem 3.2 are generalizations and extensions of Theorem 3.5 of Singh et al [38], Theorem 3 of Berinde [3], Theorem 2 of Osilike [25], Theorem 2 and Theorem 5 of Osilike and Udomene [26], Theorem 2 of Rhoades [30], Theorem 30 of Rhoades [31], Theorem 2 of Rhoades [32], Theorem 3 of Harder and Hicks [9] as well as some of the results of the author [10, 20, 21, 22].

4. Some Convergence Results in Arbitrary Banach Space

Theorem 4.1. *Let* $(E, ||.||)$ *be an arbitrary Banach space and* Y *is an arbitrary set. Suppose that* $S,\ T : Y \to E$ *are nonself operators such that* $T(Y) \subseteq S(Y),\ S(Y)$ *a complete subspace of* E, and S *is an injective operator. Let* z *be a coincidence point of* S *and* T *(that is,* $Sz = Tz = p$*). Suppose that* S *and* T *satisfy condition (2.3). Let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a monotone increasing function such that* $\varphi(0) = 0$*. For* $x_0 \in Y$*, let* $\{Sx_n\}_{n=0}^{\infty}$ *be the Jungck-Ishikawa iteration process defined by (2.1), where* $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_{\rm n}\}_{\rm n=0}^{\infty}$ *are sequences in* $[0, 1]$ *such that* $\sum_{k=0}^{\infty} \alpha_k = \infty$*. Then,* $\{Sx_n\}_{n=0}^{\infty}$ *converges strongly to* p*.*

Proof. Let $C(S, T)$ be the set of the coincidence points of S and T. We shall now use condition (2.3) to establish that S and T have a unique coincidence point z (i.e. $Sz = Tz = p$ (say)): Suppose that there exist $z_1,\ z_2 \in C(S, T)$ such that $Sz_1 = Tz_1 = p_1$ and $\rm Sz_2 = Tz_2 = p_2$. If $p_1 = p_2$, then $Sz_1 = Sz_2$ and since S is injective, it follows that $z_1 = z_2$.

If $p_1 \neq p_2$, then we have by the contractiveness condition (2.3) for S and T that

$$0 < ||p_1 - p_2|| = ||Tz_1 - Tz_2|| \leq \varphi(||Sz_1 - Tz_1||) + a||Sz_1 - Sz_2|| = a||p_1 - p_2||,$$

which leads to $(1 - a)||p_1 - p_2|| \leq 0$, from which it follows that $1 - a > 0$ since $a \in [0, 1)$, but $||p_1 - p_2|| \leq 0$, which is a contradiction since norm is nonnegative. Therefore, we have that $||p_1 - p_2|| = 0$, that is, $p_1 = p_2 = p$. Since $p_1 = p_2$, then we have that $p_1 = Sz_1 = Tz_1 = Sz_2 = Tz_2 = p_2$, leading to $Sz_1 = Sz_2 \Rightarrow z_1 = z_2 = z$ (since S is injective). Hence, $z \in C(S, T)$, that is, z is a unique coincidence point of S and T.

We now prove that $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p (where $Sz = Tz = p$) using again, condition (2.3). Therefore, we have

$$\begin{aligned}
||Sx_{n+1} - p|| &= ||(1 - \alpha_n)Sx_n + \alpha_n Tb_n - (1 - \alpha_n + \alpha_n)p|| \\
&= ||(1 - \alpha_n)(Sx_n - p) + \alpha_n(Tb_n - p)|| \\
&\leq (1 - \alpha_n)||Sx_n - p|| + \alpha_n||p - Tb_n|| \\
&= (1 - \alpha_n)||Sx_n - p|| + \alpha_n||Tz - Tb_n|| \\
&\leq (1 - \alpha_n)||Sx_n - p|| + \alpha_n[\frac{\varphi(||Sz - Tz||) + a||Sz - Sb_n||}{1 + M||Sz - Tz||}] \\
&= (1 - \alpha_n)||Sx_n - p|| + a\alpha_n||(1 - \beta_n)(p - Sx_n) + \beta_n(Tz - Tx_n)|| \\
&\leq (1 - \alpha_n)||Sx_n - p|| + a\alpha_n[\,(1 - \beta_n)||p - Sx_n|| + \beta_n||Tz - Tx_n||\,] \\
&= (1 - \alpha_n + a\alpha_n - a\alpha_n\beta_n)||Sx_n - p|| + a\alpha_n\beta_n||Tz - Tx_n|| \\
&\leq [1 - (1 - a)\alpha_n - (1 - a)a\alpha_n\beta_n]||Sx_n - p|| \\
&\leq [1 - (1 - a)\alpha_n]||Sx_n - p|| \\
&\leq \Pi_{k=0}^{n}[1 - (1 - a)\alpha_k]||Sx_0 - p|| \\
&\leq \Pi_{k=0}^{n} e^{-(1-a)\alpha_k}||Sx_0 - p|| \\
&= e^{-[(1-a)\sum_{k=0}^{n}\alpha_k]}||Sx_0 - p|| \to 0 \text{ as n} \to \infty, \qquad (4.1)
\end{aligned}$$

since $\sum_{k=0}^{\infty}\alpha_k = \infty$ and $a \in [0, 1)$. Hence, we obtain from (4.1) that $||Sx_n - p|| \to 0$ as n $\to \infty$, that is, $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. □

Theorem 4.2. *Let $(E, ||.||)$ be an arbitrary Banach space and Y is an arbitrary set. Suppose that S, $T : Y \to E$ are nonself operators such that $T(Y) \subseteq S(Y)$, $S(Y)$ a complete subspace of E,* and *S is an injective operator. Let z be a coincidence point of S and T (that is, $Sz = Tz = p$). Suppose that S and T satisfy the contractive condition (2.2). Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone increasing function such that $\varphi(0) = 0$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotone increasing function such that $\varphi(0) = 0$. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Ishikawa iteration process defined by (2.1), where $\{\alpha_n\}$* and *$\{\beta_n\}$ are sequences in $[0, 1]$ such that $\sum_{k=0}^{\infty}\alpha_k = \infty$. Then, $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.*

Proof. With $M = 0$ in the proof of Theorem 4.1, then the proof of Theorem 4.2 is completed. □

Remark 4.1. Theorem 4.1 and Theorem 4.2 are generalizations and extensions of a multitude of results. In particular, both Theorems are generalizations and extensions of both Theorem 1 and Theorem 2 of Berinde [5], Theorem 2 and Theorem 3 of Kannan [15], Theorem 3 of Kannan [16], Theorem 4 of Rhoades [33] as well as Theorem 8 of Rhoades [34]. Also, both Theorem 4 of Rhoades [33] and Theorem 8 of Rhoades [34] are Theorem 4.10 and Theorem 5.6 of Berinde [4] respectively.

Remark 4.2. In this paper, we have considered a new iteration process, namely: the Jungck-Ishikawa iteration process. This new iteration process extends the frontiers of knowledge in the fixed point theory.

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