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An approximate fixed point proof of the Browder-Göhde-Kirk fixed point theorem

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ABSTRACT. The theorem known in literature as the Theorem of Browder-Göhde-Kirk is proven by using approximate fixed point terminology and results.

1. INTRODUCTION

The classical contraction mapping principle – a fundamental result in metrical fixed point theory, first established in the case of normed spaces by Banach ([1]) – states the existence and uniqueness of the fixed point of a strict contraction in an arbitrary metric space, see for example [18], [3].

If the contractive condition is weakened - to a nonexpansiveness condition, for example, - then the existence of the fixed point is no more guaranteed. But, by giving the space a sufficiently rich geometric structure, the strict contractiveness condition on the map can be relaxed to nonexpansiveness, such that the existence (but no more the uniqueness) of the fixed point is still ensured, by the Browder-Göhde-Kirk fixed point theorem:

Theorem 1.1. ***(Browder-Göhde-Kirk)*** *Let X be a uniformly convex Banach space, Y a non-empty, bounded, closed and convex subset of X, and $T : Y \to Y$ a nonexpansive mapping. Then T has at least one fixed point in Y.*

This important result was obtained independently by F.E. Browder, D. Göhde and W.A. Kirk in 1965 (cf. [6], [10], [15]), who established existence results in Hilbert spaces, uniformly convex Banach spaces and reflexive Banach spaces, respectively. An account of the various proofs (both constructive and nonconstructive) of this, by now, classical result, may be found in Kirk [16]. An elementary proof in a uniformly convex Banach space setting was given by Goebel in 1969 [11], see also Granas and Dugundji [12], pp. 52, or Rus [18], pp. 62.

In this short note we give another proof of the Browder-Göhde-Kirk fixed point theorem, based on some well known results, such as the asymptotic regularity of the Krasnoselski averaged operator T_λ associated to the nonexpansive operator $T,$ or on some other original ones, such as the relationship between the sets of approximate or $\varepsilon-$fixed points of T and $T_\lambda,$ namely $F_\varepsilon(T)$ and $F_\varepsilon(T_\lambda).$

To this end, we need some notions, results and terminology from the approximate fixed point theory.

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2. Preliminaries

In the following we present some definitions and results used throughout this paper. We will assume $(X, \|\cdot\|)$ is a Banach space.

Definition 2.1. *A Banach space $(X, \|\cdot\|)$ is called **strictly convex** if for any x, y in X, with $x \neq y$ and $\|x\| = \|y\| = 1$, the following holds:*

$$\|\lambda x + (1-\lambda)y\| < 1, \text{ for any} \lambda \in (0,1).$$

Definition 2.2. *A Banach space $(X, \|\cdot\|)$ is called **uniformly convex** if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$ implies*

$$1/2 \|x + y\| < 1 - \delta(\varepsilon).$$

Definition 2.3. *A mapping $T : X \to X$ is called **nonexpansive** if for any $x, y \in X$*

$$\|Tx - Ty\| \leq \|x - y\|.$$

Definition 2.4. *A mapping $T : X \to X$ is called **asymptotically regular** if for any $x \in X$*

$$\left\|T^n x - T^{n+1}x\right\| \to 0, n \to \infty.$$

In 1955 M. Krasnoselski [17] proves that for any nonexpansive mapping $T : Y \to Y$ the associated ”averaged” mapping $T_{\frac{1}{2}} = 1/2(I + T)$ has its sequence of iterates converging to a fixed point of T, under the assumptions that X is a uniformly Banach space and Y is a convex compact subset of X. In 1957 H. Schaefer [20] proves the convergence of the iterates for any associated mapping $T_\lambda : Y \to Y$, $T_\lambda = \lambda I + (1-\lambda)T,$ with λ between 0 and 1 and T compact. Further on, F.E. Browder and W.V. Petryshyn prove in 1966 [7] that the uniform convexity of the space implies the asymptotic regularity of any T_λ defined on bounded convex sets.

In 1976, S. Ishikawa [13] succeeds in proving the same, but without any restriction on the geometry of the space X.

Theorem 2.1 ([13])**.** *Let $(X, \|\cdot\|)$ be a Banach space, Y a closed, bounded and convex subset of X and $T : Y \to Y$ a nonexpansive mapping. Then the Krasnoselski averaged operator associated to T, $T_\lambda : Y \to Y$, is asymptotically regular for any $\lambda \in (0,1)$.*

Considering the pioneering work of M. Krasnoselski on this topic, throughout this paper we shall use the following definition:

Definition 2.5. *Let $T : X \to X$ be a nonexpansive mapping and $\lambda \in (0,1)$. The mapping $T_\lambda : X \to X, x \longmapsto T_\lambda x = \lambda x + (1-\lambda) Tx$, for any $x \in X$, is called **the Krasnoselski averaged operator** associated to T.*

Remark 2.1. *It is easy to show that the nonexpansivity of T implies the nonexpansivity of T_λ. Besides, T and T_λ have the same set of fixed points, namely: $F(T) = F(T_\lambda)$.*

3. Nonexpansive mappings and approximate fixed points

The results included in this paragraph only require that $(X, \|\cdot\|)$ is a normed space.

Definition 3.1. *Let $(X, \|\cdot\|)$ be a normed space and $f : X \to X$ a mapping. A point $x \in X$ is called $\varepsilon-$* ***fixed point*** *or* ***approximate fixed point*** *of f if*

$$\|x - f(x)\| \le \varepsilon,$$

where $\varepsilon > 0$.

The set of $\varepsilon-$ fixed points of f is denoted by $F_\varepsilon(f)$, i.e.,

$$F_\varepsilon(f) = \{x \in X \mid \|x - f(x)\| \le \varepsilon\}.$$

Similar to the fixed point property, one may define:

Definition 3.2. *A mapping $f : X \to X$ is said to have* ***the approximate fixed point property*** *on X if for any $\varepsilon > 0$*

$$F_\varepsilon(f) \ne \emptyset.$$

The following lemma, which was proved in [2] for metric spaces, gives a sufficient condition for an operator to have the above mentioned property.

Lemma 3.1. *(*[2]*) Let $(X, \|\cdot\|)$ be a normed space and $f : X \to X$ a mapping. If f is asymptotically regular, then f has the approximate fixed point property.*

Remark 2.1 mentions a result concerning the relationship between the fixed point sets of T and T_λ. It is then natural to wonder about their approximate fixed point sets. An answer is given by:

Lemma 3.2. *Let $(X, \|\cdot\|)$ be a normed linear space, $T : X \to X$ a nonexpansive mapping and $T_\lambda : X \to X$, $\lambda \in (0,1)$ its Krasnoselski averaged operator. Then for any $\varepsilon > 0$,*

$$F_\varepsilon(T_\lambda) = F_{\frac{\varepsilon}{1-\lambda}}(T).$$

Proof. Let $\varepsilon > 0$. We shall prove the coincidence of the two sets by equivalence:

$$x \in F_\varepsilon(T_\lambda) \Leftrightarrow \|x - T_\lambda(x)\| \le \varepsilon \Leftrightarrow \|x - \lambda x - (1-\lambda)Tx\| \le \varepsilon \Leftrightarrow$$

$$\Leftrightarrow (1-\lambda)\|x - Tx\| \le \varepsilon \Leftrightarrow \|x - Tx\| \le \frac{\varepsilon}{1-\lambda} \Leftrightarrow x \in T_{\frac{\varepsilon}{1-\lambda}}(T).$$

□

It is known that if $f : Y \to X$ is a nonexpansive mapping on the convex subset Y of the strictly convex Banach space X, then its set of fixed points $F(f)$ is convex. A proof can be found in [18], where the result is included as *Theorem 8.2.1*. A similar result holds for the set of approximate fixed points:

Proposition 3.1. *Let $(X, \|\cdot\|)$ be a strictly convex Banach space, Y a convex subset of X and $T : Y \to Y$ a nonexpansive mapping. Then for any $\varepsilon > 0$, the set $F_\varepsilon(T)$ is convex.*

Proof. Let $\varepsilon > 0$ and $x^*, y^* \in F_\varepsilon(T)$. Then:

$$\|x^* - Tx^*\| \le \varepsilon, \ \|y^* - Ty^*\| \le \varepsilon.$$

We shall prove that $[x^*, y^*] \subset F_\varepsilon(T)$ as well.
Let $z^* \in (x^*, y^*)$, so $z^* = \lambda x^* + (1-\lambda)y^*$, with $\lambda \in (0,1)$.
We have:

$$\begin{aligned}
\|z^* - Tz^*\| &= \|\lambda x^* + (1-\lambda) y^* - T(\lambda x^* + (1-\lambda) y^*)\| = \\
&= \|\lambda x^* - \lambda T x^* + \lambda T x^* + (1-\lambda) y^* - \\
&\quad - (1-\lambda) T y^* + (1-\lambda) T y^* - T(\lambda x^* + (1-\lambda) y^*)\| \\
&\leq \lambda \|x^* - Tx^*\| + (1-\lambda) \|y^* - Ty^*\| + \\
&\quad + \|\lambda T x^* + (1-\lambda) T y^* - T(\lambda x^* + (1-\lambda) y^*)\| \\
&\leq \lambda\varepsilon + (1-\lambda)\varepsilon + \lambda \|Tx^* - Ty^*\| - \|T(\lambda x^* + (1-\lambda) y^*) - Ty^*\| \leq \\
&\leq \varepsilon + \lambda \|x^* - y^*\| - \|\lambda x^* + (1-\lambda) y^* - y^*\| = \\
&= \varepsilon + \lambda \|x^* - y^*\| - \lambda \|x^* - y^*\| = \varepsilon.
\end{aligned}$$

So

$$\|z^* - Tz^*\| \leq \varepsilon,$$

i.e., $z^* \in F_\varepsilon(T)$, so $F_\varepsilon(T)$ is convex for any $\varepsilon > 0$. □

For other results concerning fixed points of nonexpansive mappings see [8],[9], [14], [19].

In the following we will give a short proof of Theorem 1.1, based on the previously presented approximate fixed point terminology and results.

This proof is essentially a compilation of several ideas taken from other known proofs of Browder-Göhde-Kirk fixed point theorem, its main merit being the systematic use of concepts and results expressed in terms of approximate fixed point terminology.

***Proof of Theorem* 1.1**

We consider the Krasnoselski averaged operator associated to T,

$$T_\lambda : Y \to Y, T_\lambda = \lambda I + (1-\lambda) T, \text{ with } \lambda \in (0,1).$$

As Y is bounded, closed and convex, it follows by the theorem of Ishikawa (*Theorem 2.1*) that T_λ is asymptotically regular.

Then by the above *Lemma 3.1*, T_λ has the approximate fixed point property, that is, for any $\eta > 0$

$$F_\eta(T_\lambda) \neq \emptyset.$$

Now according to *Lemma 3.2* this implies that for any $\eta > 0$

$$F_{\frac{\eta}{1-\lambda}}(T) \neq \emptyset, \text{ where } \lambda \in (0,1).$$

But for each $\varepsilon > 0$ there exists $\eta = \varepsilon(1-\lambda)$, so that

$$F_{\frac{\varepsilon(1-\lambda)}{1-\lambda}}(T) \neq \emptyset,$$

namely for each $\varepsilon > 0$, we have that

$$F_\varepsilon(T) \neq \emptyset.$$

Now it is sufficient to prove that $\bigcap\limits_{\varepsilon>0} F_\varepsilon(T) \neq \emptyset$.

As Y is bounded, closed and convex in the uniformly convex space X, it follows that Y is weakly compact.

On the other hand T being nonexpansive it is also continuous, so $F_\varepsilon(T)$ is closed. Besides, according to *Proposition 3.1*, $F_\varepsilon(T)$ is also convex. Then $F_\varepsilon(T)$

is weakly compact for any $\varepsilon > 0$, as a closed convex subset of the weakly compact set Y.

This immediately implies that $\bigcap_{\varepsilon>0} F_\varepsilon (T) \neq \emptyset$, which completes the proof.

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