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Operators of Bernstein type

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1. INTRODUCTION

Let $m \in \mathbb{N}$ and $B_m : C([0,1]) \to C([0,1])$ be the Bernstein operators, defined for any function $f \in C([0,1])$ by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right) \tag{1.1}$$

where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k} \tag{1.2}$$

for all $x \in [0,1]$.

It is known that

$$(B_m e_0)(x) = 1\,, \quad (B_m e_1)(x) = x\,, \quad (B_m e_2)(x) = x^2 + \frac{x(1-x)}{m} \tag{1.3}$$

where $e_0(x) = 1$, $e_1(x) = x$ and $e_2(x) = x^2$, $\forall\, x \in [0,1]$, $m \in \mathbb{N}^*$.
We shall also take into account that $\varphi_x : I \to \mathbb{R}$, $\varphi_x(x) = |t - x|$ and of

Theorem 1.1. *Let $L : C(I) \to B(I)$ a linear and positive operator with the property $Le_0 = e_0$.*

(i) *If $f \in C_B(I)$, then $\forall\, x \in I, \forall\, \delta > 0$*

$$|(Lf)(x) - f(x)| \le \left(1 + \delta^{-1}\sqrt{\left(L\varphi_x^2\right)(x)}\right)\omega_f(\delta)\,. \tag{1.4}$$

(ii) *If f is derivable on I and $f' \in C_B(I)$, then $\forall\, x \in I, \forall\, \delta > 0$*

$$\begin{aligned} &|(Lf)(x) - f(x)| \le \\ &\quad \le |f'(x)|\,|(Le_1)(x) - x| + \sqrt{\left(L\varphi_x^2\right)(x)}\left(1 + \delta^{-1}\sqrt{\left(L\varphi_x^2\right)(x)}\right)\omega_{f'}(\delta)\,. \end{aligned} \tag{1.5}$$

For the proof check [4].

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2. PRELIMINARIES

We consider the sequences of nodes $\left((x_{m,k})_{k=\overline{0,m}}\right)_{m\geq 1}$ and $\left((y_{m,k})_{k=\overline{0,m}}\right)_{m\geq 1}$ defined by

$$0 = x_{m,0} < x_{m,1} < \cdots < x_{m,m-1} < x_{m,m} = 1\,, \tag{2.1}$$

$$0 = y_{m,0} < y_{m,1} < \cdots < y_{m,m-1} < y_{m,m} = 1\,, \tag{2.2}$$

$$\frac{k-1}{m} \leq x_{m,k} \leq \frac{k}{m}\,, \quad k \in \{1, 2, \ldots, m-1\}\,, \tag{2.3}$$

$$\frac{k}{m} \leq y_{m,k} \leq \frac{k+1}{m}\,, \quad k \in \{1, 2, \ldots, m-1\}\,. \tag{2.4}$$

For the sequences of nodes defined above, we define the operators sequences $(B_{m,s})_{m\geq 1}$, $(B_{m,d})_{m\geq 1}$ through

$$B_{m,s},\ B_{m,d} : C([0,1]) \to C([0,1])$$

$$(B_{m,s}f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f(x_{m,k})\,, \tag{2.5}$$

$$(B_{m,d}f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f(y_{m,k})\,, \tag{2.6}$$

where $f \in C([0,1])$ and $m \in \mathbb{N}^*$.

Proposition 2.1. *The operators from the sequences $(B_{m,s})_{m\geq 1}$, $(B_{m,d})_{m\geq 1}$ are linear and positive operators.*

Proof. Let $m \in \mathbb{N}^*$. Then we have that $\forall\ f, g \in C([0,1])$, $\forall\ \alpha, \beta \in \mathbb{R}$, $\forall\ x \in [0,1]$,

$$\begin{aligned}
\left[B_{m,s}(\alpha f + \beta g)\right](x) &= \sum_{k=0}^{m} p_{m,k}(x)(\alpha f + \beta g)(x_{m,k}) = \\
&= \sum_{k=0}^{m} p_{m,k}(x) \left[\alpha f(x_{m,k}) + \beta g(x_{m,k})\right] = \\
&= \alpha \sum_{k=0}^{m} p_{m,k}(x) f(x_{m,k}) + \beta \sum_{k=0}^{m} p_{m,k}(x) g(x_{m,k}) = \\
&= \alpha (B_{m,s}f)(x) + \beta (B_{m,s}g)(x) = \left(\alpha B_{m,s}f + \beta B_{m,s}g\right)(x)\,,
\end{aligned}$$

that $B_{m,s}$ is a linear operator.

We consider the function $f \in C([0,1])$, $f(x) \geq 0$, $\forall\ x \in [0,1]$. According to the operator's definition $B_{m,s}$, we have that $(B_{m,s}f)(x) \geq 0$, $\forall\ x \in [0,1]$, so the operator $B_{m,s}$ is positive.

Analogously we prove for $B_{m,d}$ operator. □

Proposition 2.2. *The operators $B_{m,s}$, $B_{m,d}$, $m \in \mathbb{N}^*$ verify*

$$\left(B_{m,s}e_0\right)(x) = 1, \quad \left(B_{m,d}e_0\right)(x) = 1 \tag{2.7}$$

$\forall\ x \in [0,1]$, $\forall\ m \in \mathbb{N}^*$ *and*

$$\|B_{m,s}\| = 1, \quad \|B_{m,d}\| = 1\,. \tag{2.8}$$

Proof. We have that $(B_{m,s}e_0)(x) = \sum_{k=0}^{m} p_{m,k}(x) = (B_m e_0)(x) = 1$, when we took (1.3) into account. We take into account that $\|B_{m,s}\| = \|B_{m,s}e_0\|$ and according to (2.7) $B_{m,s}e_0 = e_0$, so $\|B_{m,s}\| = 1$. Analogously, we prove for the $B_{m,d}$ operators. □

Observation 2.1. We consider $x_{m,k} = \dfrac{k}{m}$, $y_{m,k} = \dfrac{k}{m}$, $k \in \{0, 1, \dots, m\}$, and so we obtain Bernstein operators.

3. Main results

Theorem 3.2. *For all function $f \in C([0,1])$, we have*

$$\lim_{m\to\infty} (B_{m,s}f)(x) = f(x) \quad \textit{uniform on} \quad [0,1] \tag{3.1}$$

and

$$\lim_{m\to\infty} (B_{m,d}f)(x) = f(x) \quad \textit{uniform on} \quad [0,1]\,. \tag{3.2}$$

Proof. From $\dfrac{k-1}{m} \le x_{m,k} \le \dfrac{k}{m}$, by multiplying with $p_{m,k}(x)$, summing after k from 1 to $m-1$ and summing $p_{m,0}(x)x_{m,0}$ and $p_{m,m}(x)x_{m,m}$, we have

$$\sum_{k=1}^{m-1} p_{m,k}(x)\frac{k-1}{m} + x^m \le (B_{m,s}e_1)(x) \le \sum_{k=0}^{m} p_{m,k}(x)\frac{k}{m}\,,$$

or
$$\sum_{k=0}^{m} p_{m,k}(x)\frac{k}{m} - \frac{1}{m}\left[\sum_{k=0}^{m} p_{m,k}(x) - (1-x)^m - x^m\right] \le$$
$$\le (B_{m,s}e_1)(x) \le \sum_{k=0}^{m} p_{m,k}(x)\frac{k}{m}\,,$$

or $(B_m e_1)(x) - \dfrac{1}{m}\left[(B_m e_0)(x) - (1-x)^m - x^m\right] \le (B_{m,s}e_1)(x) \le (B_m e_1)(x)\,.$

Taking (1.3) into account, we have

$$x - \frac{1}{m} + \frac{1}{m}\left[(1-x)^m + x^m\right] \le (B_{m,s}e_1)(x) \le x\,,$$

or
$$x - \frac{1}{m} \le x - \frac{1}{m} + \frac{1}{m}\left[(1-x)^m + x^m\right] \le (B_{m,s}e_1)(x) \le x \le x + \frac{1}{m}\,,$$

so

$$|(B_{m,s}e_1)(x) - x| \le \frac{1}{m}\,, \quad \forall\, x \in [0,1], \quad \forall\, m \in \mathbb{N}^*\,. \tag{3.3}$$

From $\dfrac{k-1}{m} \le x_{m,k} \le \dfrac{k}{m}$, $k \in \{1, 2, \dots, m-1\}$ we have $\dfrac{(k-1)^2}{m^2} \le x_{m,k}^2 \le \dfrac{k^2}{m^2}$, $k \in \{1, 2, \dots, m-1\}$,

or
$$\sum_{k=1}^{m-1} p_{m,k}(x) \left(\frac{k^2}{m^2} - 2\frac{k}{m^2} + \frac{1}{m^2} \right) \le$$
$$\le \sum_{k=1}^{m-1} p_{m,k}(x) x_{m,k}^2 \le \sum_{k=1}^{m-1} p_{m,k}(x) \frac{k^2}{m^2}, \quad k \in \{1, 2, \dots, m-1\},$$

or
$$\sum_{k=0}^{m} p_{m,k}(x) \left(\frac{k}{m} \right)^2 - \frac{2}{m} \sum_{k=0}^{m} p_{m,k}(x) \frac{k}{m} +$$
$$+ \frac{1}{m^2} \sum_{k=0}^{m} p_{m,k}(x) - x^m + \frac{2}{m} x^m - \frac{1}{m^2} [x^m + (1-x)^m] \le$$
$$\le \sum_{k=0}^{m} p_{m,k}(x) x_{m,k}^2 - x^m \le \sum_{k=0}^{m} p_{m,k}(x) \left(\frac{k}{m} \right)^2 - x^m .$$

Thus,
$$(B_m e_2)(x) - \frac{2}{m} (B_m e_1)(x) + \frac{1}{m^2} (B_m e_0)(x) + \frac{2}{m} x^m - \frac{1}{m^2} [x^m + (1-x)^m] \le$$
$$\le (B_{m,s} e_2)(x) \le (B_m e_2)(x),$$

so
$$\frac{x(1-x)}{m} - \frac{2x}{m} + \frac{2x^m}{m} + \frac{1}{m^2} [1 - x^m - (1-x)^m] \le$$
$$\le (B_{m,s} e_2)(x) - x^2 \le \frac{x(1-x)}{m}. \quad (3.4)$$

But $x(1-x) \le \frac{1}{4}$, $\forall\, x \in [0, 1]$, so
$$0 \le \frac{x(1-x)}{m} \le \frac{1}{4m} < \frac{2}{m}, \quad \forall\, x \in [0, 1]. \quad (3.5)$$

We consider the function $g_m : [0, 1] \to \mathbb{R}$, $g_m(x) = 1 - x^m - (1-x)^m$ and we have $g_m'(x) = -m \left[x^{m-1} - (1-x)^{m-1} \right]$. If $m = 1$, the function g_1 is constant, $g_1(x) = 0$, $\forall\, x \in [0, 1]$. If $m \ge 2$, then

x	0		$\frac{1}{2}$		1
g_m'		+	0	-	
g_m	0	$\nearrow$		$\searrow$	0

so $g_m(x) \geq 0$, $\forall\, x \in [0,1]$, $\forall\, m \in \mathbb{N}^*$. Heeding (3.5) and the things proved above, we have that

$$-\frac{2}{m} \leq -\frac{2x}{m} \leq$$
$$\leq -\frac{2x}{m} + \frac{x(1-x)}{m} + \frac{2x^m}{m} + \frac{1}{m^2}\left[1 - x^m - (1-x)^m\right], \quad \forall\, x \in [0,1]. \quad (3.6)$$

From (3.4)–(3.6), it results that

$$\left|(B_{m,s}e_2)(x) - x^2\right| \leq \frac{2}{m}, \quad \forall\, x \in [0,1], \quad \forall\, m \in \mathbb{N}^*. \quad (3.7)$$

From (2.7), (3.3) and (3.7) it results that $\lim\limits_{m\to\infty}(B_{m,s}e_0)(x) = 1$, $\lim\limits_{m\to\infty}(B_{m,s}e_1)(x) = x$, $\lim\limits_{m\to\infty}(B_{m,s}e_2)(x) = x^2$ uniform on $[0,1]$ and according to the Theorem of H. Bohman and P.P. Korovkin, (3.1) follows.

In the same way, we prove that

$$|(B_{m,d}e_1)(x) - x| \leq \frac{1}{m}, \quad \forall\, x \in [0,1], \quad \forall\, m \in \mathbb{N}^* \quad (3.8)$$

and

$$\left|(B_{m,d}e_2)(x) - x^2\right| \leq \frac{3}{m}, \quad \forall\, x \in [0,1], \quad \forall\, m \in \mathbb{N}^*, \quad (3.9)$$

from where (3.2) is obtained. □

Theorem 3.3. *For all function $f \in C([0,1])$, we have that*

$$|(B_{m,s}f)(x) - f(x)| \leq 3\omega_f\left(\frac{1}{\sqrt{m}}\right), \quad \forall\, x \in [0,1], \quad \forall\, m \in \mathbb{N}^* \quad (3.10)$$

and

$$|(B_{m,d}f)(x) - f(x)| \leq$$
$$\leq \left(1 + \sqrt{5}\right)\omega_f\left(\frac{1}{\sqrt{m}}\right), \quad \forall\, x \in [0,1], \quad \forall\, m \in \mathbb{N}^*. \quad (3.11)$$

Proof. From (3.3) it results that $-\frac{1}{m} \leq (B_{m,s}e_1)(x) - x \leq \frac{1}{m}$, $\forall\, x \in [0,1]$, from where

$$-2x(B_{m,s}e_1)(x) \leq -2x^2 + \frac{2x}{m}, \quad \forall\, x \in [0,1], \quad \forall\, m \in \mathbb{N}^*. \quad (3.12)$$

From (3.7), it results that

$$(B_{m,s}e_2)(x) \leq x^2 + \frac{2}{m}, \quad \forall\, x \in [0,1], \quad \forall\, m \in \mathbb{N}^*. \quad (3.13)$$

We have that $\left(B_{m,s}\varphi_x^2\right)(x) = (B_{m,s}e_2)(x) - 2x(B_{m,s}e_1)(x) + x^2(B_{m,s}e_0)(x)$ and according to (2.7), (3.12) and (3.13), we obtain

$$\left(B_{m,s}\varphi_x^2\right)(x) \leq \frac{2(1+x)}{m} \leq \frac{4}{m}, \quad \forall\, x \in [0,1], \quad \forall\, m \in \mathbb{N}^*. \quad (3.14)$$

From the inequality (1.4), taking (3.14) into account, we have

$$|(B_{m,s}f)(x) - f(x)| \leq \left(1 + \delta^{-1}\frac{2}{\sqrt{m}}\right)\omega_f(\delta)$$

and considering $\delta = \frac{1}{\sqrt{m}}$, we obtain (3.10).

In the same way, using (2.7), (3.8) and (3.9), we obtain (3.11). □

Theorem 3.4. *If the function* $f : [0,1] \to \mathbb{R}$ *is derivable on* $[0,1]$ *and* $f' \in C([0,1])$, *then*

$$|(B_{m,s}f)(x) - f(x)| \leq \frac{1}{m}\ |f'(x)| + \frac{6}{\sqrt{m}}\,\omega_{f'}\left(\frac{1}{\sqrt{m}}\right)\ ,\ \forall\ x \in [0,1] \tag{3.15}$$

and

$$|(B_{m,d}f)(x) - f(x)|\ \leq\ \frac{1}{m}\ |f'(x)| + \frac{\sqrt{5}+5}{\sqrt{m}}\,\omega_{f'}\left(\frac{1}{\sqrt{m}}\right)\ ,\ \ \forall\ \ x\ \in\ [0,1]\,. \tag{3.16}$$

Proof. Using (1.5), (3.3) and (3.14), we have

$$|(B_{m,s}f)(x) - f(x)| \leq \frac{1}{m}\ |f'(x)| + \frac{2}{\sqrt{m}}\left(1 + \delta^{-1}\frac{2}{\sqrt{m}}\right)\omega_{f'}(\delta)$$

and considering $\delta = \frac{1}{\sqrt{m}}$, we obtain (3.15). Analogously, we obtain (3.16). □

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