CREATIVE MATH. & INF.
17 (2008), 54 - 64

The inclusion-exclusion principle and the pingenhole principle on distributive lattices

VASILE POP

ABSTRACT. We present concrete examples of lattices endowed with ”measures” (related to contest problems). The corresponding applications of the two principles are illustrated.

1. INTRODUCTION

We present concrete examples of lattices endowed with ”measures”, examples 5.1-5.6, related to contest problems. The corresponding applications of the two principles are illustrated by problems 6.1-6.7.

Usually the two principles are formulated with respect to a finite set.

If A is a finite set, $A_1, A_2, \ldots, A_n \subset A$ are subsets of A and we denote by $|X|$ the number of elements of the set $X \subset A$, then the two principles are stated as:

1. $|\bigcup\limits_{i=1}^{n} A_i| = \sum\limits_{i} |A_i| - \sum\limits_{i<j} |A_i \cap A_j| + \sum\limits_{i<j<k} |A_i \cap A_j \cap A_k| - \ldots$
   (The including-excluding principle)
2. If $\sum\limits_{i=1}^{n} |A_i| > |A|$ then there exist $i \neq j$ such that $A_i \cap A_j \neq \emptyset$.
   (Dirichlet principle)

Many problems of geometric nature require the extension of these principles; the appropriate framework is that of the distributive lattices.
On such a lattice it is necessary to replace the cardinal of a set by a suitable ”measure”.

2. PRELIMINARIES

We will give a short presentation of the notions used in this paper.

Definition 2.1. An ordered set $(L, \leq)$ is called lattice if for every x and y in L there exists

$$\inf\{x,y\} = x \wedge y \in L \text{ and } \sup\{x,y\} = x \vee y \in L. \square$$

Remark 2.1. 1) If $x \wedge y = z$ then z is defined by the properties:

$$z \leq x,\ z \leq y \text{ and if } a \leq x \text{ and } a \leq y, \text{ then } a \leq z.$$

2) If $x \vee y = u$ then u is defined by the properties:

$$x \leq u,\ y \leq u \text{ and if } x \leq a \text{ and } y \leq a, \text{ then } u \leq a.$$

Received: 15.06.2007. In revised form: 29.10.2007.
2000 *Mathematics Subject Classification*. 06D99, 05B35.
Key words and phrases. *Distributive lattice, measure, inclusion-exclusion principle, pingenhole principle, Dirichlet principle.*

3) A lattice can be regarded as a triplet $(L, \vee, \wedge)$ where "$\vee$" and "$\wedge$" are associative and commutative operations on L with the properties:

$$x \wedge (x \vee y) = x \text{ and } x \vee (x \wedge y) = x$$

for every x and y in L.

4) On the lattice $(L, \vee, \wedge)$ the order relation is defined by:

$$x \leq y \text{ iff } x \wedge y = x \text{ or iff } x \vee y = y.$$

Definition 2.2. The lattice $(L, \vee, \wedge)$ is called distributive if:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

for every x, y, z in L.

Remark 2.2. The lattice $(L, \vee, \wedge)$ is distributive iff one of the following properties holds:

a) $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ for every x, y, z in L;

b) If $x \wedge z = y \wedge z$ and $x \vee z = y \vee z$, then $x = y$.

Definition 2.3. A function $m : L \to [0, \infty]$ is called measure on the lattice $(L, \vee, \wedge)$ if $m(x \vee y) + m(x \wedge y) = m(x) + m(y)$, for every $x, y \in L$.

If $m(x \vee y) \geq m(x \wedge y)$ for every x, y in L, the measure m is called increasing measure.

Remark 2.3. 1) An increasing measure on $(L, \vee, \wedge) = (L, \leq)$ is an increasing function from the ordered set $(L, \leq)$ to the ordered set $([0, \infty], \leq)$ (if $x \leq y$ then $x \vee y = y$ and $x \wedge y = x$, hence $m(x) \leq m(y)$).

2) If the lattice L have a least element denoted by $\emptyset$ and the function take the value 0, then: $m(\emptyset) = 0$ and if $x \wedge y = \emptyset$ then $m(x \vee y) = m(x) + m(y)$.

3. The inclusion-exclusion principle

Let $(L, \vee, \wedge)$ a distributive lattice and $m : L \to [0, \infty]$ a measure on L.

Theorem 3.1. *(Inclusion-exclusion principle) For every finite set having the elements $a_1, a_2, \ldots, a_n$ in L the following relation holds:*

$$m(a_1 \vee \cdots \vee a_n) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} m(a_{i_1} \wedge \cdots \wedge a_{i_k}).$$

Proof. We prove by induction on n.

For $n = 1$, $m(a_1) = m(a_1)$ and for $n = 2$ the relation becomes

$$m(a_1 \vee a_2) = m(a_1) + m(a_2) - m(a_1 \wedge a_2)$$

obvious in the Definition 2.3. We suppose that the relation holds for n and we prove that it holds for $n + 1$.

We have:

$$m(a_1 \vee \cdots \vee a_n \vee a_{n+1}) = m((a_1 \vee \cdots \vee a_n) \vee a_{n+1}) =$$

$$m(a_1 \vee \cdots \vee a_n) + m(a_{n+1}) - m((a_1 \vee \cdots \vee a_n) \wedge a_{n+1}) =$$

$$m(a_1 \vee \cdots \vee a_n) + m(a_{n+1}) - m((a_1 \wedge a_{n+1}) \vee \cdots \vee (a_n \wedge a_{n+1})) =$$

$$\sum_{k=1}^{n}(-1)^{k+1}\sum_{1\le i_1<\cdots<i_k\le n} m(a_{i_1}\wedge\cdots\wedge a_{i_k})+m(a_{n+1})-$$

$$-\sum_{k=1}^{n}(-1)^{k+1}\sum_{1\le i_1<\cdots<i_k\le n} m(a_{i_1}\wedge\cdots\wedge a_{i_k}\wedge a_{n+1})=$$

$$\sum_{i_1=1}^{n} m(a_{i_1}+m(a_{n+1})+\sum_{k=2}^{n}(-1)^{k+1}\sum_{1\le i_1<\cdots<i_k\le n} m(a_{i_1}\wedge\cdots\wedge a_{i_k})-$$

$$-(-1)^{n-1}m(a_1\wedge\cdots\wedge a_n\wedge a_{n+1})-\sum_{k=1}^{n-1}(-1)^{k+1}\sum_{1\le i_1<\cdots<i_k\le n} m(a_{i_1}\wedge\cdots\wedge a_{i_k}\wedge a_{n+1})=$$

$$(-1)^{n+1}\sum_{i_1=1}^{n+1} m(a_{i_1})+\sum_{k=2}^{n+1}(-1)^{k+1}\sum_{1\le i_1<\cdots<i_k\le n+1} m(a_{i_1}\wedge\cdots\wedge a_{i_k})+$$

$$+\sum_{k=2}^{n+1}(-1)^{k+1}\sum_{1\le i_1<\cdots<i_k\le n+1} m(a_{i_1}\wedge\cdots\wedge a_{i_k})+(-1)^{n+2}m(a_1\wedge\cdots\wedge a_1\wedge a_{n+1})=$$

$$\sum_{k=1}^{n+1}(-1)^{k+1}\sum_{1\le i_1<\cdots<i_k\le n+1} m(a_{i_1}\wedge\cdots\wedge a_{i_k}).$$

□

Corollary 3.1. *(Dual principle) For every $a_1, a_2, \ldots, a_n$ in L we have:*

$$m(a_1\wedge\cdots\wedge a_n)=\sum_{k=1}^{n}(-1)^{k+1}\sum_{1\le i_1<\cdots<i_k\le n} m(a_1\vee\cdots\vee a_{i_k}).$$

Proof. Using the symmetry of ”$\vee$” and ”$\wedge$” in the definition of measure and the distributivity of the lattice we may replace in Theorem 3.1 ”$\vee$” by ”$\wedge$”. □

Remark 3.1. 1) If $A=\{a_i|\, i\in I\}$ with I a finite set, then Theorem 3.1 and Corollary 3.1 become:

$$T_1:\quad m\left(\bigvee_{i\in I} a_i\right)=\sum_{K\subset I}(-1)^{|K|+1} m\left(\bigwedge_{k\in K} a_k\right)$$

$$C_1:\quad m\left(\bigwedge_{i\in I} a_i\right)=\sum_{K\subset I}(-1)^{|K|+1} m\left(\bigvee_{k\in K} a_k\right)$$

where $|K|$ is the number of elements of the nonempty set K.

2) If we denote by

$$m_k=\sum_{1\le i_1<\cdots<i_k\le n} m(a_{i_1}\wedge\cdots\wedge a_{i_k})$$

and

$$m_0=m\left(\bigvee_{i=1}^{n} a_i\right)$$

then from Theorem 3.1 holds:

$$\sum(-1)^k m_k = 0.$$

4. THE DIRICHLET PRINCIPLE

We deal in what follows with Dirichlet principle.

Let $(L, \vee, \wedge)$ be a distributive lattice and $m : L \to [0, \infty]$ a measure on L. For $a_1, a_2, \ldots, a_n \in L$ we denote

$$I_k = \bigvee_{1 \le i_1 < \cdots < i_k \le n} (a_{i_1} \wedge \cdots \wedge a_{i_k}), \quad k = \overline{1, n}$$

$$(I_1 = \bigvee_{i=1}^{n} a_i,\ I_2 = \bigvee_{1 \le i < j \le n} (a_i \wedge a_j), \ldots, I_n = \bigwedge_{i=1}^{n} a_i). \square$$

Theorem 4.1. *(Pingen hole principle or Dirichlet principle) For every finite set with elements $a_1, a_2, \ldots, a_n$ in L we have the relation:*

$$\sum_{k=1}^{n} m(a_k) = \sum_{k=1}^{n} m(I_k).$$

Proof. We prove by induction on n.

For $n = 1$ the relation becomes $m(a_1) = m(a_1)$, and for $n = 2$

$$m(a_1) + m(a_2) = m(a_1 \vee a_2) + m(a_1 \wedge a_2),$$

which is true by Definition 2.3.

For $n + 1$ we have

$$I'_k = \bigvee_{1 \le i_1 < \cdots < i_k \le n+1} (a_{i_1} \wedge \cdots \wedge a_{i_k}) = I_k \vee (a_{n+1} \wedge I_{k-1}),$$

hence:

$$m(I'_k) = m(I_k) + m(a_{n+1} \wedge I_{k-1}) - m(I_k \wedge a_{n+1})$$

and

$$\sum_{k=1}^{n} m(I'_k) = \sum_{k=1}^{n} m(I_k) + m(a_{k+1}) - m(a_{n+1} \wedge I_n) =$$

$$\sum_{k=1}^{n} m(a_k) + (a_{k+1}) - m(I'_{n+1}),$$

then

$$\sum_{k=1}^{n+1} m(I'_k) = \sum_{k=1}^{n+1} m(a_k).$$

$\square$

Corollary 4.1. *(Dual relation) If we denote $U_k = \bigwedge_{1 \le i_1 < \cdots < i_k \le n} (a_{i_1} \vee \cdots \vee a_{i_k})$ then:*

$$\sum_{k=1}^{n} m(a_k) = \sum_{k=1}^{n} m(U_k).$$

Let $(L, \vee, \wedge)$ be a distributive lattice with the least element $\emptyset$ and let m be an increasing measure on L with $m(\emptyset) = 0$. In these conditions the following consequences of Theorem 4.1 holds:

Corollary 4.2. *If* $a_{i_1} \wedge \cdots \wedge a_{i_p} \wedge a_{i_{p+1}} = \emptyset$ *for every different numbers* $i_1, \ldots, i_p, i_{p+1} \in \{1, 2, \ldots, n\}$ *then:*

$$\sum_{k=1}^{n} m(a_k) \leq p \cdot m\left(\bigvee_{k=1}^{n} a_k\right).$$

Proof. The condition implies $I_{p+1} = \emptyset$. From $I_n \leq I_{n-1} \leq \cdots \leq I_{p+1} \leq \cdots \leq I_1$ follows

$$m(I_n) \leq m(I_{n-1}) \leq \cdots \leq m(I_{p+1}) \leq m(I_p) \leq \cdots \leq m(I_1)$$

and hence

$$\sum_{k=1}^{n} m(a_k) = \sum_{k=1}^{n} m(I_k) = m(I_1) + \cdots + m(I_p) \leq pm(I_1).$$

□

Corollary 4.3. *If* $p \in \{1, 2, \ldots, n\}$ *and*

$$\sum_{k=1}^{n} m(a_k) > p \cdot m\left(\bigvee_{k=1}^{n} a_k\right)$$

then there exist different $i_1, i_2, \ldots, i_p, i_{p+1}$ *such that:*

$$a_{i_1} \wedge \cdots \wedge a_{i_p} \wedge a_{i_{p+1}} \neq \emptyset.$$

Proof. If $a_{i_1} \wedge \cdots \wedge a_{i_p} \wedge a_{i_{p+1}} = \emptyset$ for every $i_1 < \cdots < i_{p+1}$ then $I_{p+1} = \emptyset$ and from Corollary 4.2, the contrary inequality holds. □

Corollary 4.4. *If* $p \in \{1, 2, \ldots, n-1\}$ *and*

$$\sum_{k=1}^{n} m(a_k) > p \cdot m\left(\bigvee_{k=1}^{n} a_k\right)$$

then there exist different $i_1, \ldots, i_p, i_{p+1}$ *such that:*

$$m(a_{i_1} \wedge \cdots \wedge a_{i_p} \wedge a_{i_{p+1}}) \geq \frac{D}{(n-p)\dbinom{n}{p+1}}$$

where $D = \sum_{k=1}^{n} m(a_k) - p \cdot m\left(\bigvee_{k=1}^{n} a_k\right)$.

Proof. From Theorem 4.1 we have

$$\sum_{k=1}^{n} m(a_k) = m(I_1) + \cdots + m(I_p) + m(I_{p+1}) + \cdots + m(I_n) \leq$$

$$\leq pm(I_1) + (n-p)m(I_{p+1}) = p \cdot m\left(\bigvee_{k=1}^{n} a_k\right) + (n-p)m(I_{p+1}),$$

hence

$$m(I_{p+1}) \geq \frac{D}{n-p}.$$

But

$$m(I_{p+1}) = m\left(\bigvee_{1\leq i_1<\cdots<i_{p+1}\leq n}(a_{i_1}\wedge\cdots\wedge a_{i_{p+1}})\right) \leq$$

$$\leq \sum_{1\leq i_1<\cdots<i_{p+1}\leq n} m(a_{i_1}\wedge\cdots\wedge a_{i_{p+1}}).$$

This sum contains $\binom{n}{p+1}$ terms, so then there exists one of them such that

$$m(a_{i_1}\wedge\cdots\wedge a_{i_{p+1}}) \geq \frac{D}{(n-p)\binom{n}{p+1}}.$$

$\square$

Remark 4.1. *If $m(a_1)+\cdots+m(a_n) > m(a_1\vee\cdots\vee a_n)$ then there exist $i,j\in\{1,2,\ldots,n\}$, $i\neq j$ such that $a_i\wedge a_j\neq\emptyset$.*

This is a particular case of Corollary 4.3 (for $p=1$) frequently used as pingenhole principle.

5. Examples of distributive lattices with measure

Example 5.1. $((M),\subset) = ((M),\cup,\cap)$, $m(X)=|X|$.

If M is a finite set and (M) is the family of all subsets of M, (M) together with inclusion relation is a distributive lattice in which $X\vee Y = X\cup Y$ and $X\wedge Y = X\cap Y$, for every $X,Y\subset M$.

The function $m:(M)\to\mathbb{N}$, $m(X)=|X|=$ the number of elements of the set $X\subset M$, is an increasing measure on (M). The lattice (M) has the least element the empty set $\emptyset$.

If M is an infinite set, then ${}_F(M)$, the set of all finite subsets of M, is a distributive lattice and on ${}_F(M)$ can be considered the same measure.

Example 5.2. $(\mathbb{R},\leq) = (\mathbb{R},\max,\min)$, $m(x)=|x|$.

The set $\mathbb{R}$ of real numbers, with the usual order relation "$\leq$" is a distributive lattice in which:

$$x\vee y = \max\{x,y\} = \frac{x+y+|x-y|}{2}$$
$$x\wedge y = \min\{x,y\} = \frac{x+y-|x-y|}{2}$$

The function $m:\mathbb{R}\to[0,\infty)$, $m(x)=|x|$ is a measure on $\mathbb{R}$ ($|\max\{x,y\}| = |x|+|y|-|\min\{x,y\}|$). Because $\mathbb{R}$ has not a least element and the measure is not increasing on $\mathbb{R}$, we can use only Theorem 3.1, Theorem 3.2 and Corollary 3.1 but we cannot apply Corollary 4.1, 4.2 and 4.3.

As sublattices of the lattice we have the lattices $(\mathbb{Z},\max,\min)$, $(\mathbb{N},\max,\min)$ and $(\mathbb{R}_+,\max,\min)$.

The second and the third has the least element 0 and the measure $m(x) = x$ is an increasing measure. We can use Corollary 4.1, 4.2 and 4.3.

Example 5.3. $(\mathbb{N}^*, |) = (\mathbb{N}^*, l.c.m., g.c.d)$, $m(x) = \log x$.

The set of positive integers $\mathbb{N}^* = \{1, 2, 3, \dots\}$ with the divisibility relation, is a distributive lattice $(\mathbb{N}^*, |)$ in which:

$$x \vee y = [x, y] = l.c.m.\{x, y\}$$

$$x \wedge y = (x, y) = g.c.d.\{x, y\}.$$

The function $m : \mathbb{N}^* \to [0, \infty)$, $m(x) = \log x$ (the base of logarithm is a number higher than 1) verifies the relation

$$\log([x, y]) + \log((x, y)) = \log x + \log y,$$

that is increasing on $\mathbb{N}^*$ and $m(1) = 0$, 1 is the least element. In this example Theorem 3.1 and Corollary 3.1 are:

$$[a_1, a_2, \dots, a_n] = \frac{\prod\limits_{i=\overline{1,n}} a_i \prod\limits_{i_1<i_2<i_3} (a_{i_1}, a_{i_2}, a_{i_3}) \dots}{\prod\limits_{i_1<i_2} (a_{i_1}, a_{i_2}) \prod\limits_{i_1<i_2<i_3<i_4} (a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}) \dots}$$

At nominator we have all greatest common divisors of the subsets with odd numbers of elements and at the denominator we have all greatest common divisors of subsets with an even number of elements from the set $\{a_1, a_2, \dots, a_n\}$.

$$(a_1, a_2, \dots, a_n) = \frac{\prod a_{i_1} \prod [a_{i_1}, a_{i_2}, a_{i_3}] \dots}{\prod [a_{i_1}, a_{i_2}] \prod [a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}] \dots}$$

Example 5.4. A probability space $(\mathcal{P}, \cup, \cap)$ with a random function $p : \to [0, 1]$ is a particular case of the lattice $((M), \cup, \cap)$.

Example 5.5. $(\mathbb{R}_u[X], |)$, $m(f) = deg(f)$.

If $\mathbb{R}_u[X]$ is the set of all unitary polynomials with real coefficients, the divisibility relation is an order relation and $(\mathbb{R}_u[X], |)$ is a distributive lattice in which:
$f \vee g = [f, g]$, the least common multiple and
$f \wedge g = (f, g)$, the greatest common divisor. The function $m : \mathbb{R}_u[X] \to \mathbb{N}$, $m(f) = deg(f)$ is an increasing measure and $f = 1$ is the least element.

Example 5.6. $(J_n, \cup, \cap)$, the set of all Jordan-measurable sets from $\mathbb{R}^n$, and $m(X)$ is the Jordan measure of X, which is an increasing measure.

The measure in $\mathbb{R}^n$ of the parallelepiped

$$P = \prod_{i=1}^{n} [a_i, b_i]$$

is

$$m(P) = \prod_{i=1}^{n} (b_i - a_i).$$

In particular case the measure in $\mathbb{R}$ (on a line) is the length, the measure in $\mathbb{R}^2$ (on a plane) is the area, the measure in $\mathbb{R}^3$ (in the space) is the volume. Because $m(\emptyset) = 0$ all theoretical results that we have obtained may be applied.

6. Applications

In this section we will give some applications.

Problem 6.1. Prove that for every 1000 points situated in a disc with the radius $R = 1$, there exists a disc with the radius $r = \dfrac{1}{9}$ which cover at least 11 points.

Solution. We prove that 11 of the discs D_i, $i = \overline{1, 1000}$ with the radius $r = \dfrac{1}{9}$, centered in the given points have the intersection nonempty. The maximum covered surface is less than that of the given disc, with the radius increased to $1 + \dfrac{1}{9}$, so

$$S\left(\bigcup_{i=1}^{1000} D_i\right) < \pi\left(1 + \frac{1}{9}\right)^2 = \frac{100\pi}{81}.$$

On the other hand, taking into account Corollary 4.3.

$$\sum_{i=1}^{1000} S(D_i) = 1000\frac{\pi}{81},$$

so

$$\sum_{i=1}^{1000} S(D_i) > 10S\left(\bigcup_{i=1}^{1000} D_i\right)$$

Problem 6.2. Consider the natural numbers a, b, c and denote by M their least common multiple. Prove that if $abc > 4M$, then two of the numbers a, b, c have a common divisor $d \geq 3$.

Solution. From the third example we have

$$M = [a, b, c] = \frac{abc(a, b, c)}{(a, b)(b, c)(c, a)}.$$

From $abc > 4M$ it follows that $4(a, b, c) < (a, b)(b, c)(c, a)$.

If we suppose $(a, b) \leq 2$, $(b, c) \leq 2$ and $(c, a) \leq 2$, then two situations can hold:

1. If $(a, b) = (b, c) = (c, a) = 2$, then $(a, b, c) = 2$ and the inequality becomes $4 \cdot 2 < 2 \cdot 2 \cdot 2$ (false).

2. If $(a, b) = 1$ then $(a, b, c) = 1$ and the inequality becomes

$$4 < (b, c)(c, a) \leq 2 \cdot 2 = 4 \quad \text{(false)}$$

So there exists $d \in \{(a, b), (b, c), (c, a)\}$ with $d \geq 3$.

Problem 6.3. Consider $a_1, a_2, \ldots, a_n \in \mathbb{N}^*$ and let M be their least common multiple. Prove that if $a_1 a_2 \ldots a_n > M^p$ for some $p \in \mathbb{N}^*$, then $n \geq p + 1$ and there exist $p + 1$ numbers $a_{i_1}, a_{i_2}, \ldots, a_{i_{p+1}}$ which have a common divisor $d \geq 2$.

Solution. From Example 5.3 it follows that $(\mathbb{N}^*, (\cdot,\cdot), [\cdot,\cdot])$ is a distributive lattice and the function $\lg : \mathbb{N}^* \to [0,\infty)$ is a measure. Taking the logarithm in the given relation we obtain

$$\sum_{i=1}^{n} \lg a_i > p \lg M$$

and using Corollary 4.3 it follows that there exist $a_{i_1}, a_{i_2}, \ldots, a_{i_{p+1}}$ such that

$$(a_{i_1}, a_{i_2}, \ldots, a_{i_{p+1}}) \neq 1.$$

Problem 6.4. In the interior of a polygon having the area 13 we take 10 polygons having the area 6. Prove that there exist 4 polygons having the overlapping surface greater than $\dfrac{1}{70}$.

Solution. The area is a measure of the polygons in the plane (see Example 5.6). If P_i, $i = \overline{1,10}$ are the ten polygons, we have

$$S\left(\bigcup_{i=1}^{10} P_i\right) \leq 13 \text{ and } \sum_{i=1}^{10} S(P_i) = 60,$$

so the difference

$$D = \sum_{i=1}^{10} S(P_i) - 3S\left(\bigcup_{i=1}^{10} P_i\right)$$

is positive.

Using Corollary 4.4 we obtain that there exist the polygons $P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}$ such that

$$S(P_{i_1} \cap P_{i_2} \cap P_{i_3} \cap P_{i_4}) \geq \frac{D}{7C_{10}^4} = \frac{21}{7 \cdot 210} = \frac{1}{70}.$$

Problem 6.5. In a cube with the edge 1 we consider n spheres with the sum of their areas 32. Prove that there exists a line which intersects at least 9 spheres.

Solution. Denoting by S_k the area of the diametral circle of the k^{th} sphere we know that

$$\sum_{k=1}^{n} 4S_k = 32.$$

Because $S_k \leq \dfrac{\pi}{4}$ (the area of the circle with the radius $\dfrac{1}{2}$) we obtain $4n\dfrac{\pi}{4} > 32$, so $n > 10$.

If we consider the projections D_k, of the spheres S_k onto one of the faces of the given cube, we will have

$$\sum_{k=1}^{n} S_k = 8 > 8S\left(\bigcup_{k=1}^{n} D_k\right)$$

and from Corollary 4.3 it follows that there exist 9 discs having nonempty intersection. The perpendicular to this plane which passes through one of the points of intersection of the 9 discs, intersects the 9 spheres (those which have been projected).

Problem 6.6. Let $A_1,\dots,A_n$ be finite sets and let $x_1,\dots,x_n$ be integer numbers with the sum: $x_1+\cdots+x_n=0$.

Show that
$$\sum_{i=1}^n\sum_{j=1}^n |A_i\cup A_j|x_ix_j\le 0.$$
($|A|$ is the number of elements of A).

Solution. $|A_i\cup A_j|=|A_i|+|A_j|-|A_i\cap A_j|$.
$$\sum_{i=1}^n\sum_{j=1}^n |A_i\cup A_j|x_ix_j=\sum_{i=1}^n\sum_{j=1}^n(|A_i|+|A_j|)x_ix_j-$$
$$-\sum_{i=1}^n\sum_{j=1}^n |A_i\cap A_j|x_ix_j=2\left(\sum_{k=1}^n x_k\right)\left(\sum_{k=1}^n x_k|A_k|\right)-$$
$$-\sum_{i=1}^n\sum_{j=1}^n |A_i\cap A_j|x_ix_j=-\sum_{i=1}^n\sum_{j=1}^n |A_i\cap A_j|x_ix_j.$$

If $\bigcup_{k=1}^n A_k=\{a_1,a_2,\dots,a_N\}$ we assign to every set A_k a vector $V_k=(v_{k1},\dots,v_{kN})$ where:
$$v_{ki}=\begin{cases}1 & \text{if}\quad a_i\in A_k\\ 0 & \text{if}\quad a_i\notin A_k\end{cases}$$
and thus
$$|A_i\cap A_j|=\sum_{i=1}^N v_{ik}v_{jk}.$$

We have
$$-\sum_{i=1}^n\sum_{j=1}^n |A_i\cap A_j|x_ix_j=-\sum_{k=1}^N\left(\sum_{i=1}^n\sum_{j=1}^n(x_iv_{ik})(x_jv_{jk})\right)=$$
$$=-\sum_{k=1}^N(x_1v_{1k}+\cdots+x_nv_{nk})^2\le 0.$$

Problem 6.7. Let N be a natural number and $a_1,a_2,\dots,a_n$ some natural divisors of it (not necessarily distinct). Prove that
$$S=\sum_{1\le i_1\le n}a_{i_1}-\sum_{1\le i_1<i_2\le n}(a_{i_1},a_{i_2})+\sum_{1\le i_1<i_2<i_3\le n}(a_{i_1},a_{i_2},a_{i_3})-\cdots+(-1)^{n-1}(a_1,a_2,\dots,a_n)$$
is a natural number less than or equal to N. Characterize the situation when $S=N$. ($(x_1,\dots,x_k)$ is the greatest common divisor of the numbers $x_1,\dots,x_k$).

Solution. If $a_1,\dots,a_k$ are divisors of N then we have:
$$\left(\frac{N}{a_1},\frac{N}{a_2},\dots,\frac{N}{a_k}\right)[a_1,a_2,\dots,a_k]=N$$
(This can be proved by induction on k).

Let $b_i = \dfrac{N}{a_i}$ and $A_i = \{b_i, 2b_i, \dots, a_i b_i = N\}$. A_i is the set of the multiples of b_i; it has $|A_i| = \dfrac{N}{b_i} = a_i$ elements, $i = \overline{1,n}$. Then

$$|A_{i_1} \cap A_{i_2}| = \frac{N}{[b_{i_1}, b_{i_2}]} = (a_{i_1}, a_{i_2})$$

$$|A_{i_1} \cap A_{i_2} \cap A_{i_3}| = \frac{N}{[b_{i_1}, b_{i_2}, b_{i_3}]} = (a_{i_1}, a_{i_2}, a_{i_3})$$

and we have

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{1\le i_1\le n} |A_{i_1}| - \sum_{1\le i_1<i_2\le n} |A_{i_1} \cap A_{i_2}| + \sum_{1\le i_1<i_2<i_3\le n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \dots =$$

$$= \sum a_{i_1} - \sum (a_{i_1}, a_{i_2}) + \sum (a_{i_1}, a_{i_2}, a_{i_3}) - \dots$$

Each of the sets A_i contains only numbers from the set $\{1, \dots, N\}$. So

$$\left|\bigcup_{i=1}^{n} A_i\right| \le N.$$

Suppose that the numbers $a_1, \dots, a_k$ are less than N. Then $b_1, \dots, b_k$ are different from 1 and the sets A_{i_k} contain only multiples of $b_i \ge 2$. Thus $\bigcup\limits_{i=1}^{n} A_i$ does not contain prime numbers, hence $\left|\bigcup\limits_{i=1}^{n} A_i\right| < N$.

This means that $S = N$ if and only if at least one of the numbers $a_1, \dots, a_n$ is N.

References

[1] Berge, C., *Principes de combinatoire*, Dunod, Paris, 1968

[2] Breslauer, D., Dubhashi, D.P., *Combinatorics for Computer Scientists*, Brics Lecture Series, LS-95-4, 1995

[3] Pimsner, M., Popa, S., *Problems in elementary geometry*, E.D.P. Bucureşti, 1979

[4] Purdea, I., Pic, Gh., *Treatise of modern algebra*, E.A. Bucureşti, 1977

[5] Ryser, H., *Combinatorial mathematics*, John Wiley, New York, 1963

[6] Tomescu, I., *Introduction to combinatorics*, E.Th. Bucureşti, 1972

Department of Mathematics
Technical University of Cluj-Napoca
C. Daicoviciu 15
400020 Cluj-Napoca, Romania
E-mail address: Vasile.Pop@math.utcluj.ro