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A sharp criterion for the univalence of the Libera operator

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ABSTRACT. Let f be an analytic function of the form $f(z) = z + a_2z^2 + ...$ defined in the unit disc $U = \{z \in C : |z| < 1\}$. Suitable values of β have been determined by a number of authors, so that $\mathrm{Re}\big(zf''(x) + \frac{z^2}{3}f'''(z)\big) > -\beta,\ z \in U$ implies the starlikeness of f. In all these previous papers the method of differential-subordination has been used. We improve their results using the method of convolution and obtain the biggest possible value of β so that the differential inequality $\mathrm{Re}\big(zf''(x) + \frac{z^2}{3}f'''(z)\big) > -\beta,\ z \in U$ implies the univalence of the function f. The integral version of the result involving Libera operator is given.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of analytic functions defined in the unit disk

$$U = \{z \in C : |z| < 1\}$$

which have the development $f(z) = z + a_2z^2 + ...$ Let K be the subclass of $\mathcal{A}$ which consists of the convex functions. It is well-known that $f \in K$ is equivalent to

$$Re(1 + \frac{zf''(z)}{f'(z)}) > 0, z \in U.$$

The integral operator of Libera will be denoted by

$$L(f)(z) = \frac{2}{z}\int_0^z f(t)dt.$$

The authors proved in [1] and in [2, p. 279] the following result:
[*Corollary 5.2 d.1.]* If $f \in \mathcal{A}$ and

$$Re(zf''(z)) > -\frac{3}{7}, \quad z \in U$$

then $L(f) \in K$.

This Corollary shows that the function $L(f)$ is univalent in U, provided that $Re(zf''(z)) > -\frac{3}{7}, z \in U$. The problem to determine the biggest $c > 0$, for which the inequality $Re(zf''(z)) > -c, z \in U$ implies the univalence of $L(f)$ in U, comes up naturally.
We are going to answer this question below, but at first we recall some known results, which we need in our study.

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2. Preliminaries

Definition 2.1. [3] *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $g(z) = \sum_{n=0}^{\infty} b_n z^n$ *be two analytic functions in* U*. The convolution of the functions* f *and* g *is defined by the equality*

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Let A_0 be the the class of analytic functions in U which satisfy $f(0) = 1$.

Definition 2.2. [3] *If* $V \subset A_0$ *then the dual of V denoted by* V^d *consists of functions* g *which satisfy* $g \in A_0$ *and* $(f * g)(z) \neq 0$ *for every* $f \in V$ *and every* $z \in U$*.*

The class $\mathcal{P}$ is the subset of A_0 defined by

$$\mathcal{P} = \{f \in A_0 : Re(f(z)) > 0, z \in U\}.$$

Lemma 2.1. [3], p. 23 *(Duality theorem) The dual of the class* $\mathcal{P}$ *is*

$$\mathcal{P}^d = \{f \in A_0 | Re(f(z)) > \frac{1}{2},\ z \in U\}.$$

Let h_T be the function defined by the equality

$$h_T(z) = \frac{1}{1+iT}\left[iT\frac{z}{1-z} + \frac{z}{(1-z)^2}\right],\ \ T \in \mathbb{R}.$$

It is simple to observe that h_T is an element of the class $\mathcal{A}$.

Lemma 2.2. [3], p. 94 *The function* $f \in \mathcal{A}$ *belongs to* S^**, the class of the starlike functions if and only if* $\frac{f(z)}{z} * \frac{h_T(z)}{z} \neq 0$ *for all* $T \in R$ *and for all* $z \in U$*.*

Lemma 2.3. [3] (The Herglotz formula) For all $f \in \mathcal{P}$ there exists a probability measure μ on the interval $[0, 2\pi]$, so that

$$f(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t)$$

or in developed form

$$f(z) = 1 + 2\int_0^{2\pi} \left(\sum_{n=1}^{\infty} z^n e^{-in}\right) d\mu(t).$$

The converse of the theorem is also valid.

3. The main result

Theorem 3.1. *The biggest value of* c *for which the condition*

$$Re\left(zf''(z) + \frac{z^2}{3}f'''(z)\right) > -c,\ \ z \in U \tag{3.1}$$

implies the starlikeness of the function f *is* $c = \frac{2}{3}$*.*

Proof. The condition (3.1) implies that

$$\frac{c + zf''(z) + \frac{z^2}{3}f'''(z)}{c} \in \mathcal{P}$$

and from the Herglotz representation theorem we get that there is a measure μ so that

$$f(z) = z + 6c\sum_{n=2}^{\infty} \frac{z^n}{n(n-1)(n+1)} \int_0^{2\pi} e^{-i(n-1)t} d\mu(t),\ \mu([0,2\pi]) = 1.$$

Lemma 2.2 implies that $f \in S^*$ if and only if

$$\frac{f(z)}{z} * \frac{h_T(z)}{z} \neq 0, z \in U,\ T \in \mathbb{R} \quad \text{where}\ h_T(z) = z + \sum_{n=1}^{\infty} \frac{n+1+iT}{1+iT} z^{n+1}.$$

The condition of starlikeness becomes after some calculations

$$\begin{aligned}\frac{f(z)}{z} * \frac{h_T(z)}{z} &= \left(1 + 6c\sum_{n=2}^{\infty} \frac{z^{n-1}}{n(n-1)(n+1)} \int_0^{2\pi} e^{-i(n-1)t} d\mu(t)\right) * \\ &* \left(1 + \sum_{n=1}^{\infty} \frac{n+1+iT}{1+iT} z^n\right) = \left(1 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int} d\mu(t)\right) * \\ &* \left(1 + 3c\sum_{n=1}^{\infty} \frac{n+1+iT}{(1+iT)(n+1)(n+2)n} z^n\right) \neq 0, z \in U, T \in \mathbb{R}.\end{aligned}$$

The Lemma 2.1 implies that the condition for starlikeness holds, if and only if

$$Re(1 + 3c\sum_{n=1}^{\infty} \frac{n+1+iT}{(1+iT)(n+1)(n+2)n} z^n) > \frac{1}{2}, z \in U,\ T \in \mathbb{R}.$$

or equivalently

$$Re\left(\frac{1}{6c} + \sum_{n=1}^{\infty} \frac{n+1+iT}{(1+iT)(n+1)(n+2)n} z^n\right) > 0,\ z \in U,\ T \in \mathbb{R}. \tag{3.2}$$

We are going to determine

$$m = \inf_{\substack{z\in U\\ T\in R}} Re(\frac{1}{6c} + \sum_{n=1}^{\infty} \frac{n+1+iT}{(1+iT)(n+1)(n+2)n} z^n)$$

and from $m = 0$ we will get the value of c. The maximum principle implies that

$$m = \inf_{\substack{\theta\in(0,2\pi)\\ T\in\mathbb{R}}} Re(\frac{1}{6c} + \sum_{n=1}^{\infty} \frac{n+1+iT}{(1+iT)(n+1)(n+2)n} e^{in\theta}).$$

It is simple to observe that

$$\begin{aligned}Re\left(\sum_{n=1}^{\infty} \frac{n+1+iT}{(1+iT)(n+1)(n+2)n} e^{in\theta}\right) &= \frac{1}{1+T^2}\left(T^2 \sum_{n=1}^{\infty} \frac{\cos n\theta}{n(n+1)(n+2)} + \right.\\ &\left. + T\sum_{n=1}^{\infty} \frac{\sin n\theta}{(n+1)(n+2)} + \sum_{n=1}^{\infty} \frac{\cos n\theta}{n(n+2)}\right).\end{aligned}$$

We will use the following equalities:

$$\sum_{n=1}^{\infty}\frac{e^{in\theta}}{n(n+2)}=\frac{1}{2}\int_0^1(1-t^2)\frac{e^{i\theta}-t}{1+t^2-2t\cos\theta}dt$$
$$\sum_{n=1}^{\infty}\frac{e^{in\theta}}{(n+1)(n+2)}=\int_0^1(t-t^2)\frac{e^{i\theta}-t}{1+t^2-2t\cos\theta}dt$$
$$\sum_{n=1}^{\infty}\frac{e^{in\theta}}{n(n+1)(n+2)}=\frac{1}{2}\int_0^1(1-t)^2\frac{e^{i\theta}-t}{1+t^2-2t\cos\theta}dt. \tag{3.3}$$

Let $M(\theta,T)$ be defined by the equality

$$M(\theta,T)=\frac{1}{1+T^2}\left(T^2\sum_{n=1}^{\infty}\frac{\cos n\theta}{n(n+1)(n+2)}+T\sum_{n=1}^{\infty}\frac{\sin n\theta}{(n+1)(n+2)}+\sum_{n=1}^{\infty}\frac{\cos n\theta}{n(n+2)}\right).$$

It results from (3.3) that

$$M(\theta,T)=\frac{1}{1+T^2}\Bigg(T^2\frac{1}{2}\int_0^1(1-t)^2\frac{\cos\theta-t}{1+t^2-2t\cos\theta}dt+$$
$$+T\int_0^1(t-t^2)\frac{\sin\theta}{1+t^2-2t\cos\theta}dt+\frac{1}{2}\int_0^1(1-t^2)\frac{\cos\theta-t}{1+t^2-2t\cos\theta}dt\Bigg)$$

and this leads to

$$\begin{aligned}M(\theta,T) &= -\frac{1}{2}\int_0^1(1-t)dt+\frac{1}{1+T^2}\Bigg[T^2(\frac{1}{2}\int_0^1(1-t)^2\frac{\cos\theta-t}{1+t^2-2t\cos\theta}dt+\\ &+\int_0^1\frac{(1-t)^2}{1+t}dt)+\frac{T^2}{2}(\int_0^1(1-t)dt-\int_0^1\frac{(1-t)^2}{1+t}dt)+\\ &+T\int_0^1(t-t^2)\frac{\sin\theta}{1+t^2-2t\cos\theta}dt+\\ &+\frac{1}{2}(\int_0^1(1-t^2)\frac{\cos\theta-t}{1+t^2-2t\cos\theta}dt+\int_0^1(1-t)dt)\Bigg].\end{aligned}$$

We can rewrite $M(\theta,T)$ as follows:

$$\begin{aligned}M(\theta,T) &= -\frac{1}{4}+\frac{1}{1+T^2}\Bigg(\frac{T^2(1+\cos\theta)}{2}\int_0^1\frac{(1-t)^3}{(1+t)(1+t^2-2t\cos\theta)}dt+\\ &+T^2(\frac{3}{2}-2\ln 2)+T\sin\theta\int_0^1\frac{t(1-t)}{1+t^2-2t\cos\theta}dt+\\ &+\frac{1}{2}(1+\cos\theta)\int_0^1\frac{(1-t)^2}{1+t^2-2t\cos\theta}dt\Bigg).\end{aligned} \tag{3.4}$$

Let L_1 and L_2 be

$$L_1(\theta,T)=\frac{T^2(1+\cos\theta)}{2}\int_0^1\frac{(1-t)^3}{(1+t)(1+t^2-2t\cos\theta)}dt \tag{3.5}$$

and

$$L_2(\theta,T) = T^2(\frac{3}{2}-2\ln 2)+T\sin\theta\int_0^1 \frac{t(1-t)}{1+t^2-2t\cos\theta}dt+ \\ +\frac{1}{2}(1+\cos\theta)\int_0^1 \frac{(1-t)^2}{1+t^2-2t\cos\theta}dt. \tag{3.6}$$

From (3.4), (3.5) and (3.6) we deduce

$$M(\theta,T) = -\frac{1}{4}+\frac{1}{1+T^2}\big(L_1(\theta,T)+L_2(\theta,T)\big). \tag{3.7}$$

It is simple to observe that

$$L_1(\pi,0)=0, \quad L_2(\pi,0)=0, \quad L_1(\theta,T)\geq 0 \quad \text{for } \theta\in(0,2\pi),\ T\in\mathbb{R} \tag{3.8}$$

and we will also prove that $L_2(\theta,T)\geq 0, \theta\in(0,2\pi), T\in R$. L_2 is a polynomial with respect to T and has the discriminant $\Delta_2(\theta)$:

$$\Delta_2(\theta) = 4\cos^2\frac{\theta}{2}\Bigg(\bigg(\Big(\int_0^1 \frac{t(1-t)\sin\frac{\theta}{2}}{1+t^2-2t\cos\theta}dt\Big)^2-\Big(\frac{3}{2}-2\ln 2\Big)\int_0^1 \frac{(1-t)^2}{1+t^2-2t\cos\theta}dt\Bigg).$$

Since $\int_0^1 \frac{t^2}{(1+t)^2}dt = \frac{3}{2}-2\ln 2$ we get that

$$\Delta_2(\theta) = 4\cos^2\frac{\theta}{2}\Bigg(\Big(\int_0^1 \frac{t(1-t)\sin\frac{\theta}{2}}{1+t^2-2t\cos\theta}dt\Big)^2-\int_0^1 \frac{t^2}{(1+t)^2}dt\int_0^1 \frac{(1-t)^2}{1+t^2-2t\cos\theta}dt\Bigg).$$

The inequality of Cauchy-Buniakowski-Schwarz implies:

$$\int_0^1 \frac{t^2}{(1+t)^2}dt\int_0^1 \frac{(1-t)^2}{1+t^2-2t\cos\theta}dt \geq \left(\int_0^1 \frac{t(1-t)}{(1+t)\sqrt{1+t^2-2t\cos\theta}}dt\right)^2. \tag{3.9}$$

The inequality

$$\int_0^1 \frac{t(1-t)}{(1+t)\sqrt{1+t^2-2t\cos\theta}}dt \geq \int_0^1 \frac{t(1-t)|\sin\frac{\theta}{2}|}{1+t^2-2t\cos\theta}dt \tag{3.10}$$

is valid if

$$\frac{t(1-t)}{(1+t)\sqrt{1+t^2-2t\cos\theta}} \geq \frac{t(1-t)|\sin\frac{\theta}{2}|}{1+t^2-2t\cos\theta}, \theta\in(0,2\pi), t\in[0,1]. \tag{3.11}$$

The inequality (3.11) holds, because it is equivalent to

$$(1-t)^2(1+\cos\theta)\geq 0, \theta\in(0,2\pi), t\in[0,1].$$

Now combining (3.9), (3.10) follows $\Delta_2(\theta)\leq 0, \theta\in(0,2\pi)$, and this leads to

$$L_2(\theta,T)\geq 0, \theta\in(0,2\pi), T\in R. \tag{3.12}$$

From (3.7), (3.8) and (3.12) we deduce

$$m = \inf_{\substack{\theta\in(0,2\pi)\\ T\in R}} Re(\frac{1}{6c}+\sum_{n=1}^{\infty}\frac{n+1+iT}{(1+iT)(n+1)(n+2)n}e^{in\theta}) = \frac{1}{6c}+\inf_{\substack{\theta\in(0,2\pi)\\ T\in R}} M(\theta,T) = \\ \frac{1}{6c}-\frac{1}{4}+\inf_{\substack{\theta\in(0,2\pi)\\ T\in R}}\frac{1}{1+T^2}(L_1(\theta,T)+L_2(\theta,T)) = \frac{1}{6c}-\frac{1}{4}.$$

We get the biggest value of c from the equation $m = 0$. The obtained result is sharp. $\square$

Corollary 3.1. *Let*

$$F(z) = L(f)(z) = \frac{2}{z}\int_0^z f(t)dt. \tag{3.13}$$

The biggest value of c, for which the condition $Re(zf''(z)) > -c$, $z \in U$ implies $F \in S^*$, is $c = 1$.

Proof. From the definition of F we get $zF''(z) + \frac{z^2}{3}F'''(z) = \frac{2}{3}zf''(z)$ and so the condition $Re(zf''(z)) > -c,\ z \in U$ implies

$$Re(zF''(z) + \frac{z^2}{3}F'''(z)) > -\frac{2}{3},\ z \in U.$$

Consequently we can apply Theorem 3.1. to obtain $F \in S^*$. $\square$

Theorem 3.2. *If $0 < c \leq 1$ then the condition*

$$Re(zf''(z)) > -c,\ z \in U \tag{3.14}$$

implies the univalence of the function F defined by (3.13). For $c > 1$ there exists f satisfying (3.14) so that the corresponding function F is not univalent in U.

Proof. According to the previous corollary the function F is starlike and also univalent in U. As we saw in the proof of Theorem 3.1, the condition of the theorem implies that

$$F(z) = z + 4c\sum_{n=2}^{\infty}\frac{z^n}{n(n-1)(n+1)}\int_0^{2\pi} e^{-i(n-1)t}d\mu(t),\ z \in U,\ \mu([0, 2\pi]) = 1.$$

It is well-known, that if the function F is univalent, then $F'(z) \neq 0, \forall z \in U$. A simple calculation leads to

$$F'(z) = \left(1 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int}d\mu(t)\right) * \left(1 + 2c\sum_{n=1}^{\infty}\frac{z^n}{n(n+2)}\right), z \in U.$$

Since $1 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int}d\mu(t) \in \mathcal{P}$, Definition 2.2 says that the condition $F'(z) \neq 0,\ z \in U$ holds true if and only if

$$1 + 2c\sum_{n=1}^{\infty}\frac{z^n}{n(n+2)} \in \mathcal{P}^d.$$

According to Lemma 2.1 this is equivalent to

$$Re\left(1 + 2c\sum_{n=1}^{\infty}\frac{z^n}{n(n+2)}\right) > \frac{1}{2}, z \in U. \tag{3.15}$$

We observe that

$$(1-t^2)\frac{\cos\theta - t}{1 + t^2 - 2t\cos\theta} \geq -\frac{1-t^2}{1+t},\ \ \theta \in [0, 2\pi],\ t \in [0, 1]$$

and equality holds if and only if $\theta = \pi$. This observation implies

$$\inf_{z\in U} Re\left(1+2c\sum_{n=1}^{\infty}\frac{z^n}{n(n+2)}\right) = \inf_{\theta\in(0,2\pi)} Re\left(1+2c\sum_{n=1}^{\infty}\frac{e^{in\theta}}{n(n+2)}\right) =$$

$$= \inf_{\theta\in(0,2\pi)} Re\left(1+c\int_0^1(1-t^2)\frac{\cos\theta - t}{1+t^2-2t\cos\theta}\right) = 1-c\int_0^1\frac{1-t^2}{1+t}dt = 1-\frac{c}{2}.$$

The deduced equality means that (3.15) holds true if and only if

$$1-\frac{c}{2}\geq\frac{1}{2}, \quad \text{which is equivalent to} \quad 1\geq c.$$

□

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