CREATIVE MATH. & INF.
17 (2008), 72 - 77

Generalization of classification trees for a poset

LAURA VERES

ABSTRACT. The present article studies directed trees and classification trees defined in a partially ordered set. In the first chapter we recall the notion of the classification trees for a poset, and the basic notions for our investigations. In the second chapter we analyse the relations between classification trees and tolerance classes, we present a new construction of classification trees, and also we discuss the relation between classification trees and orthogonal systems.

1. INTRODUCTION

First, we recall some definitions which were introduced in article [7]:

Definition 1.1. ([7]) Let $(P, \leq)$ be a non-empty partially ordered set (poset):

(i) P is called a meet-semilattice if for any subset of P with two elements exists their infimum;

(ii) P is a lattice if the supremum and the infimum of any two elemented subset of P there exists in P. P is called a complete lattice if the supremum and the infimum exists for any subset of P;

(iii) P is called bounded if it has a largest element 1 and a smallest element 0.

Definition 1.2. ([7]) Let $(P, \leq)$ be a poset with the largest element $1 \in P$ and H a non-empty subset of P, then

(i) H is a directed forest, if $[x) \cap H$ is a chain for any $x \in H$; (1)

(ii) a directed forest H is called directed tree if $1 \in H$. (2)

Definition 1.3. ([7]) Let $(P, \leq)$ be a bounded poset and $H \subseteq P$ a directed tree in $(P, \leq)$. H is a classification tree, if $(H \cup \{0\}, \leq_{H \cup \{0\}})$ is a meet-semilattice. (3)

H is a maximal classification tree, if there is no other classification tree which contains it as a proper subset.

Proposition 1.1. *In every poset the meet of any non-empty family of directed forests is also a directed forest.*

Proof. Let $H_j, j \in J$ be a set of directed forests and $x \in \bigcap_{j \in J} H_j$. Then $[x) \cap (\bigcap_{j \in J} H_j) = \bigcap_{j \in J} ([x) \cap H_j)$ is a chain (the meet of chains is empty or is a chain). It can not be empty, because always contains x. □

We will introduce a relation ρ as follows:

For all $a, b \in H$ we have $a \rho b \Leftrightarrow a \leq b$ or $b \leq a$ or $a \wedge b = 0$.

Received: 5.06.2007. In revised form: 23.11.2007.
2000 *Mathematics Subject Classification.* 06A06, 06A23, 06B05, 06B15, 06B99.
Key words and phrases. *Classification trees, tolerance classes, context, concept lattices, Dedekind-MacNeille completion, orthogonal systems.*

It is obvious that the relation ρ is reflexive and symmetric, therefore it is a tolerance relation.

Definition 1.4. ([6]) Let $(P, \leq)$ be a poset, $B \subseteq P$ a subset and ρ a tolerance relation on P, then

(i) B is tolerance preclass, if for each $x, y \in B$ we have $x\rho y$;

(ii) B is a tolerance class, if B is maximal for the previous property, i.e. for all $z \notin B$ exists $b \in B$ such that $(z, b) \notin \rho$;

(iii) the set $C_\rho = \{c \in P | c\rho a$ for all $a \in P\}$ is called the tolerance center of P.

It is well known (see [6]), that the tolerance center is the meet of all tolerance classes, i.e. if K_i is a set of ρ tolerance classes, then $C_\rho = \bigcap_{i\in I} K_i$.

We present a relation between classification trees and tolerance preclass proved in paper [7] and we give a better proof of it.

Proposition 1.2. ([7]) *Let $(P, \leq)$ be a bounded poset and $H \subseteq P$ a subset containing 1. H is a classification tree if and only if H is a ρ tolerance preclass.*

Proof. " $\Longrightarrow$ " : We have to prove that any two elements of H are in relation ρ.

Take $a, b \in H$. If $a \wedge b = 0$, then we get $a\rho b$.

If $a \wedge b = c \neq 0$, then $a, b \geq c$ and this implies $a, b \in [c) \cap H$. Now, in view of property (1), a and b must be comparable. So, we get again $a\rho b$.

" $\Longleftarrow$ " : Conversely, assume that H is a ρ tolerance preclass with the largest element 1. We have to prove the three properties of the classification trees.

(2) is satisfied since $1 \in H$.

(1) Let $h \in H$. In order to prove that $[h) \cap H = \{c \in H | c \geq h\}$ is a chain, take some $c_1, c_2 \in H$ with $c_1, c_2 \geq h$. Then $c_1 \wedge c_2 \neq 0$. As H is a ρ tolerance preclass we have also $c_1 \rho c_2$. Hence, by the definition of ρ we obtain $c_1 \geq c_2$ or $c_2 \geq c_1$.

(3) For all $a, b \in H$ we have $a\rho b$. In view of the definition of ρ we get
$a \wedge b \in \{a, b, 0\} \subseteq H \cup \{0\}$. Hence $H \cup \{0\}$ is a $\wedge$-semilattice.

Finally, we obtain that H is a classification tree. □

We present an other relation between classification trees and tolerance classes in the following theorem:

Theorem 1.1. *Let $(P, \leq)$ be a bounded poset. A non-empty subset $H \subseteq P$ is a ρ tolerance class if and only if H is a maximal classification tree.*

Proof. " $\Longrightarrow$ " : Let H be a ρ tolerance class. Since $h\rho 1$ holds for all $h \in H$, we get $1 \in H$. Hence using Proposition 1.2, we obtain that H is a classification tree.

We will prove by contradiction that H is a maximal classification tree.

If H is not a maximal classification tree, then there exists a classification tree $K \subseteq P$, with $H \subseteq K$, $K \neq H$. H is a ρ tolerance class, in view of Proposition 1.2 K is a ρ tolerance preclass and also $H \subseteq K$, $K \neq H$. As this is a contradiction, we get that H is a maximal classification tree.

" $\Longleftarrow$ " : Assume that H is a maximal classification tree. In view of Proposition 1.2 H is a ρ tolerance preclass. We will prove, that H is a tolerance class, i.e. it is a maximal tolerance preclass. Suppose that $H \subseteq K$, $K \neq H$ for a tolerance class K. Then K is a classification tree. As H is a maximal classification tree, $H \subseteq K$, $K \neq H$ is a contradiction. □

2. CLASSIFICATION TREES AND ORTHOGONAL SYSTEMS

A context (see [2],[4],[5]) is a triple (G, M, I) where G and M are sets and $I \subseteq G \times M$ is a binary relation. The elements of G and M are called objects and respectively attributes of the context. The relation $gI\ m$ means that the object g has the attribute m. A small context can be easily represented by a cross table, i.e., by a rectangular table, the rows of which are headed by the object names and the columns are headed by the attribute names. A cross in the intersection of the row g and the column m, means that the object g has the attribute m.

For all sets $A \subseteq G$ and $B \subseteq M$ we define

$A' = \{m \in M \mid g\ I\ m \text{ for all } g \in A\}$,

$B' = \{g \in G \mid g\ I\ m \text{ for all } m \in B\}$.

A concept of the context (G, M, I) is a pair (A, B) , in which $A' = B$ and $B' = A$, and $A \subseteq G$, $B \subseteq M$. (G, M, I) denotes the set of all concepts of the context (G, M, I).

(G, M, I) can be endowed with structure of a complete lattice defining the join and meet of concepts as follows:

$$\bigwedge_{i\in I}(A_i, B_i) = \left(\bigcup_{i\in I} A_i, \left(\bigcap_{i\in I} B_i\right)''\right)$$

$$\bigvee_{i\in I}(A_i, B_i) = \left(\left(\bigcap_{i\in I} A_i\right)'', \bigcup_{i\in I} B_i\right)$$

The lattice $((G, M, I), \wedge, \vee)$ is called the *concept lattice* of the context (G, M, I).

Definition 2.5. ([1]) (i) Let $(P, \leq)$ be a poset and $h : P \to P$ a map. The map h is called closure operator if it is monotone, extensive and idempotent.

(ii) A closure system on a set A is a set of subsets which contains A and is closed under intersections. Formally, $\Phi \subseteq \wp(A)$ is a closure system, if $A \in \Phi$ and $S \subseteq \Phi$ implies $\cap S \in \Phi$.

(iii) Let $(A, \leq)$ be a poset. For $X \subseteq A$ we will denote with X^* and X_* the set of all upper and lower bounds of X. A closure system of the closure operator $\varphi : \wp(A) \to \wp(A)$, $X \to (X^*)_*$ is called Dedekind-MacNeille completion of the poset $(A, \leq)$.

Let $(P, \leq)$ be a poset , $(P, P, \leq)$ a corresponding context and $(P, P, \leq)$ the concept lattice of this context. It is well known, that the concept lattice $(P, P, \leq)$ is the Dedekind-MacNeille completion of the poset $(P, \leq)$. So there exists an embedding

$\varphi : (P, \leq) \to ((P, P, \leq), \leq)$ such that

$a \leq b \Leftrightarrow \varphi(a) \leq \varphi(b)$ and $\varphi(x) = ((x], [x))$ for all $x \in P$.

Theorem 2.2. *Let $(P, \leq)$ be a bounded poset and $H \subseteq P$ a classification tree, then $\varphi(H)$ is a classification tree in $(P, P, \leq)$.*

Proof. We have to prove, that $\varphi(H)$ satisfies the three properties of the classification trees.

(2) It is known that $\varphi(1)$ is the largest element in $(P, P, \leq)$. As $1 \in H$, we get $\varphi(1) \in \varphi(H)$.

(1) Let $a = ((x], [x)) \in \varphi(H)$, then $a = \varphi(x)$ for some $x \in H$. We prove that $[a) \cap \varphi(H)$ is a chain (which means, that for any elements $b, c \in \varphi(H)$, $b, c \geq a$ we have $b \leq c$ or $c \leq b$).

Take some elements $b, c \in \varphi(H)$. Then $\exists\, u, v \in H$ such that $b = ((u], [u))$, $c = ((v], [v))$ and $b, c \geq a$. As $((u], [u)) \geq ((x], [x))$, $((v], [v)) \geq ((x], [x))$, we get $(x] \leq (u]$ and $(x] \leq (v]$. Then $x \leq u$ and $x \leq v$. As $u, v \in [x) \cap H$, we have $u \leq v$ or $v \leq u$ and this implies that $(u] \leq (v]$ or $(v] \leq (u]$ holds. Then $b = ((u], [u)) \leq ((v], [v)) = c$ or $c = ((v], [v)) \leq ((u], [u)) = b$.

(3) We have to prove that $(\varphi(H) \cup \varphi(0), \leq)$ is a $\wedge$-semilattice of the lattice $(P, P, \leq)$.

Take $a, b \in \varphi(H) \cup \varphi(0)$, then exists three cases:

(i) If $a = (\{0\}, P) = \varphi(0)$ or $b = (\{0\}, P) = \varphi(0)$, ($\varphi(0)$ is the smallest element of the lattice), then we get $a \wedge b = \varphi(0)$.

(ii) Let $a, b \in \varphi(H)$ with $a \leq b$ or $b \leq a$, then $a \wedge b = \inf\{a, b\} \in \{a, b\}$.

(iii) Let $a, b \in \varphi(H)$ incomparable. Then there exist $x, y \in H$ such that $a = ((x], [x))$, $b = ((y], [y))$ and x,y are incomparable. As H is a classification tree, in view of Proposition 1.2 we get $x\rho y$. As x,y are incomparable, then in view of the definition of the relation ρ, we have $x \wedge y = 0$.

Hence $a \wedge b = ((x], [x)) \cap ((y], [y)) = ((x \wedge y], [x \wedge y)) = (\{0\}, P) = \varphi(0) \in \varphi(H) \cup \varphi(0)$.

Summarizing the above cases, we obtain that $\varphi(H) \cup \varphi(0)$ is a $\wedge$-semilattice. $\square$

Definition 2.6. ([3]) Let L be a lattice with the smallest element 0. A set $O = \{a_i | \, i \in \mathbb{I}\}$, $I \neq \varnothing$ of nonzero elements of L is called an orthogonal system, if $a_i \wedge a_j = 0$, for all $i \neq j$, $i, j \in I$.

O is a maximal orthogonal system, if there is no other orthogonal system O' of L containing O as a proper subset.

Definition 2.7. Let L be a lattice with the smallest element 0 and $S_1 = \{a_i | i \in I\}$, $I \neq \varnothing$ and $S_2 = \{b_j | j \in J\}$, $J \neq \varnothing$ two orthogonal systems. We write $S_1 \leq S_2$, if for each $i \in I$ there exists $j(i) \in J$ such that $a_i \leq b_{j(i)}$. It is easy to see that $\leq$ is a partial order.

Remark 2.1. Observe that $S_0 = \{1\}$ is the greatest orthogonal system of L.

Definition 2.8. A lattice L has a finite height, if the length of any chains of it is less or equal then a fixed number $l > 0$.

Remark 2.2. It is easy to see that in the case of a lattice with finite height any subposet $(A, \leq)$ of it has the following property: for any $a \in A$ there exists a maximal element m_a in $(A, \leq)$ such that $a \leq m_a$.

Now we are prepared to prove our main theorem:

Theorem 2.3. Let L be a lattice with finite height and $H \subseteq L$ a non-empty subset of it. Then $H = \bigcup\limits_{\lambda \in \Lambda} S_\lambda$, where $\{S_\lambda | \, \lambda \in \Lambda\}$ is a chain of orthogonal systems containing $S_0 = \{1\}$ if and only if H is a classification tree.

Proof. " $\Longrightarrow$ " : (2) Clearly $1 \in H$, (so H has a largest element 1).

(1) Take an element $a \in H$ and any elements $b, c \in [a) \cap H$. Then $a \in S_{\lambda_0}$, $b \in S_{\lambda_1}$, $c \in S_{\lambda_2}$ for some $\lambda_0, \lambda_1, \lambda_2 \in \Lambda$ and clearly, $a \leq b, c$ implies $S_{\lambda_0} \leq S_{\lambda_1}$ and $S_{\lambda_0} \leq S_{\lambda_2}$.

Since $S_\lambda, \lambda \in \Lambda$ is a chain, S_{λ_1} and S_{λ_2} must be comparable. Without lost of generality we can assume $S_{\lambda_1} \leq S_{\lambda_2}$.

We will prove by contradiction that b and c are comparable.

Indeed, assume that b and c are incomparable. Since $S_{\lambda_1} \leq S_{\lambda_2}$ there exists an element $d \in S_{\lambda_2}$ such that $b \leq d$. Since $d \neq c$ and S_{λ_2} is an orthogonal system, we get $b \wedge c \leq d \wedge c = 0$ which is a contradiction, because of $b \wedge c \geq a \neq 0$. Thus we obtain that b and c are comparable.

(3) Take $x, y \in H \cup \{0\}$. If x or y is 0, then $x \wedge y = 0 \in H \cup \{0\}$. Suppose that $x, y \neq 0$, then $x, y \in H$. If x and y are comparable, then $x \wedge y \in \{x, y\} \subseteq H$.

Now, assume that x, y are incomparable elements of H. In view of the definition of H there exists $\lambda, \lambda' \in \Lambda$ such that $x \in S_\lambda$, $y \in S_{\lambda'}$. Since $\{S_\lambda | \lambda \in \Lambda\}$ is a chain, without lost of generality we can assume $S_\lambda \leq S_{\lambda'}$. Hence there exists an element $z \in S_{\lambda'}$ with $x \leq z$. As x, y are incomparable we must have $z \neq y$. Then $x \wedge y \leq z \wedge y = 0$, since $S_{\lambda'}$ is an orthogonal system. Hence $x \wedge y = 0 \in H \cup \{0\}$.

Thus we obtain $x \wedge y \in H \cup \{0\}$ in each of the possible cases.

" $\Longleftarrow$ " : Conversely, assume that H is a classification tree. Then in view of Proposition 1.2, H is a ρ tolerance preclass.

Now, we construct systems of elements S_n, $n = 0, 1, ...$ as follows: $S_0 = \{1\}$, $S_1 = \left\{a_i^{(1)} | i \in I^{(1)}\right\}$ consists of maximal elements of $H \setminus \{S_0\}$ (i.e. of all elements $a_i^{(1)}$ which satisfy $a_i^{(1)} \prec 1$)

... $S_{n+1} = \left\{a_i^{(n+1)} | i \in I^{(n+1)}\right\}$ contains all maximal elements of the set $H \setminus (S_0 \cup S_1 \cup ... \cup S_n)$.

We prove by induction on n that $S_0 \geq S_1 \geq ... \geq S_n$ is a chain of orthogonal systems.

Let $n = 1$. Clearly, $S_0 = \{1\}$ is an orthogonal system. As the elements of S_1 are maximal in $H \setminus \{S_0\}$ they must be incomparable. Since H is a ρ tolerance preclass, for all $a_i, a_j \in S_1, i \neq j$ we get $a_i \wedge a_j = 0$. Hence S_1 is an orthogonal system. Clearly, $S_0 = \{1\} \geq S_1$.

Now, assume that our hypothesis holds for $n = k$, and let us prove that it holds for $n = k + 1$ too.

First, we observe that the set $H \setminus (S_0 \cup S_1 \cup ... \cup S_k)$ is also a ρ tolerance preclass and $S_{k+1} \subseteq H \setminus (S_0 \cup S_1 \cup ... \cup S_{k-1})$. Since S_k contains maximal elements of $H \setminus (S_0 \cup S_1 \cup ... \cup S_{k-1})$, for any $a_i^{(k+1)} \in S_{k+1}$ there exists an $a_l^{(k)} \in S_k$ such that $a_i^{(k+1)} < a_l^{(k)}$.

Now, take any elements $a_i^{(k+1)} a_j^{(k+1)} \in S_{k+1}$, $i \neq j$. Then there exists two cases:

- there exists an element $a_l^{(k)} \in S_k$ such that $a_i^{(k+1)}, a_j^{(k+1)} < a_l^{(k)}$. As $a_i^{(k+1)}, a_j^{(k+1)}$ are incomparable and $\left(a_i^{(k+1)}, a_j^{(k+1)}\right) \in \rho$, we obtain $a_i^{(k+1)} \wedge a_j^{(k+1)} = 0$, proving that the elements are orthogonal.

- if $a_i^{(k+1)}, a_j^{(k+1)}$ has no common upperbound in S_k, by construction there exists

$a_m^{(k)}, a_p^{(k)} \in S_k$, $m \neq p$, $m, p \in I^{(k)}$ such that $a_i^{(k+1)} < a_m^{(k)}$ and $a_j^{(k+1)} < a_p^{(k)}$. As by induction hypothesis $a_m^{(k)} \wedge a_p^{(k)} = 0$ we get $a_i^{(k+1)} \wedge a_j^{(k+1)} \leq a_m^{(k)} \wedge a_p^{(k)} = 0$ and this implies that $S_{k+1} = \left\{ a_i^{(k+1)} | i \in I^{(k+1)} \right\}$ is an orthogonal system.

Observe, that in both of above cases $S_k \geq S_{k+1}$. Hence $S_0 \geq S_1 \geq ... \geq S_k \geq S_{k+1}$ is a chain. Thus we proved by induction that $S_0 \geq S_1 \geq ... \geq S_k \geq S_{k+1} \geq ...$ is a chain of orthogonal systems.

Observe that this chain of systems of orthogonal elements also contains at least one decreasing chain of elements: $1 > a_{i_1}^{(1)} > a_{i_2}^{(2)} > ... > a_{i_{k+1}}^{(k+1)} > ...$

As in H the length of any chain is less or equal then l, the construction process ends in at most l-steps, let say in $q \leq l$, $q \in \mathbb{N}$ steps.

Then $H^* = H \setminus (S_0 \cup S_1 \cup ... \cup S_q) = \varnothing$, otherwise H^* has maximal elements and we can continue the process. Therefore, we obtain $H = {}_{n=0}^{q} S_n$. □

REFERENCES

[1] Czédli, G., *Hálóelmélet*, Szeged, 1996

[2] Ganter, B., Wille, R., *Formal Concept Analysis*, Mathematical Foudations. Springer Verlag, Berlin Heidelberg New York (1999)

[3] Körei, A. and Radeleczki, S., *Box elements in a concept lattice*, Contribution to ICFCA 2006, Editors Bernhard Ganter, Leonard Kwuida, Verlag Allgemeine Wissenschaft, Drezda 2006 February 12-17, p. 41-55

[4] Radeleczki, S., *Classification systems and their lattice*, Discussiones Mathematicae, General Algebra and Applications 22 (2002), p. 167-181

[5] Radeleczki, S., Tóth T., *Fogalomhálók alkalmazása a csoporttechnológiában, Osztályozó rendszerek matematikai alapjainak összehasonlító vizsgálata*, 2. Kutat ási jelentés Miskolc, 2001 március

[6] Ju.A. Srejder, *Egyenlőség, hasonlóság, rendezés*, Gondolat (1975), Budapest

[7] Vereş, Laura, *Classification trees and orthogonal systems*,International Scientific Conference microCAD2007, Miskolc, 2007, Vol G, p. 63-68

UNIVERSITY OF MISKOLC
INSTITUTE OF MATHEMATICS
MISKOLC-EGYETEMVÁROS
3515 HUNGARY
E-mail address: lauracicu@yahoo.com