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The shape of Earth-type planets

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ABSTRACT. This paper deals with the application of the equation of the surface of a rotating massive body for the surface of the Earth-type planets. A contradiction with the present situation appears, but having a geological explanation.

1. THE SPACE-TIME METRICS IN THE NEIGHBOURHOOD OF A ROTATING MASSIVE BODY

It is easy to show [1] that in the geometrized spherical coordinates

$$x^0 = t = ct_{ph};\ \ x^1 = r = r_{ph};\ \ x^2 = \theta = \theta_{ph};\ \ x^3 = \varphi = \varphi_{ph} \tag{1}$$

$$c = 2,99792458 \cdot 10^8 \frac{m}{s} = \text{ speed of Light; [2] and [3]}$$

$$\varphi = \begin{cases} \phi + \omega t, \quad \text{with } \dfrac{d\varphi}{dt} = \omega \quad \text{(inside the body)} \\ \phi, \qquad \text{with } \dfrac{d\varphi}{dt} = 0 \quad \text{(outside the body),} \end{cases} \tag{2}$$

the space-time metrics, in the neighbourhood of a body in stationary rotation, is

$$ds^2 = g_{00}\left(dx^0\right)^2 + g_{11}\left(dx^1\right)^2 + g_{22}\left(dx^2\right)^2 + g_{33}\left(dx^3\right)^2 + 2g_{03}dx^0dx' \tag{3}$$

where

$$g_{ij} = g_{ij}\left(x^1, x^2\right).$$

The index ”ph” shows the fact that the respective magnitude is measured in physical units, as if measured by a remote observer.

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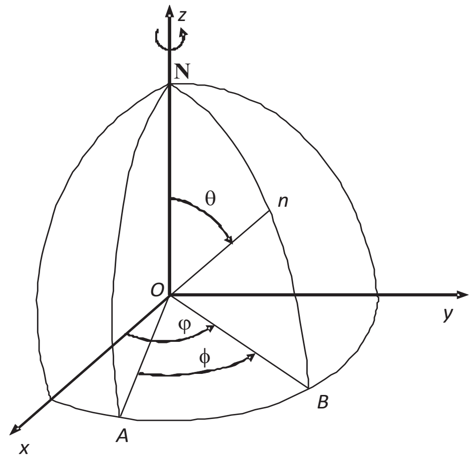

We define the quadrivelocity

$$\mathbf{u}(u^0, u^1, u^2, u^3) \tag{4}$$

where

$$u^i = \frac{dx^i}{ds} \tag{5}$$

In our case

$$u^0 = \left(g_{00} + 2\omega g_{03} + \omega^2 g_{33}\right)^{-\frac{1}{2}}; \quad u^1 = u^2 = 0; \;\; u^3 = \omega u^0, \tag{6}$$

where ω represents the angular speed of the central body, compared to the fixed reference frame round the axis Cz.

When the central body is rotating, the space-time in its exterior is characterized by Kerr's metrics [2]

$$ds^2 = \frac{\Delta}{\sum}\left(dt + a\sin^2\theta d\phi\right)^2 - \frac{\sin^2\theta}{\sum}\left[\left(r^2+a^2\right)d\phi - a\,dt\right]^2 - \frac{\sum}{\Delta}dr^2 - \sum d\theta^2 \tag{7}$$

where the following notations have been employed

$$\Delta = r^2 - 2mp + a^2; \quad \sum = r^2 + a^2\cos^2\theta. \tag{8}$$

Knowing that the universal gravitational constant [3]

$$G = 6.6732 \cdot 10^{-11}\frac{m^3}{kg\,s^2} \tag{9}$$

the relations between the geometrized magnitudes and the physical ones are

$$\begin{array}{lll} \text{length} & l = l_{ph} & (m) \\ \text{time} & t = ct_{ph} & (m) \\ \text{mass} & m = \dfrac{G}{c^2} M_{ph} & (m) \\ \text{angular momentum} & J = \dfrac{G}{c^3} J_{ph} & (m) \\ \text{specific angular momentum} & a = \dfrac{a_{ph}}{c} = \dfrac{J}{m} & (m) \\ \text{angular speed} & \omega = \dfrac{\omega_{ph}}{c} & (m^{-1}) \end{array} \tag{10}$$

2. The surface-equation of a rotating massive body

If we note by

$$\varepsilon = \rho c^2 \tag{11}$$

the density of energy and by P the pressure inside the rotating central body then the equation of the hydrostatic equilibrium is under the form

$$\frac{dP}{\varepsilon + P} = d \ln u^0 \tag{12}$$

We define the surface of the body the place where the pressure and the density become null, where

$$u^0 = u_s^0 = \text{constant} \tag{13}$$

For a non-rotating body, $\omega = 0,\ a = 0$, from

$$u_s^0 = \left(1 - \frac{2m}{R}\right)^{-\frac{1}{2}} = \text{constant} \tag{14}$$

it results R=constant, namely the body is spherical.

For a stationary rotation the equation of the intersection between the surface of the rigid body and a plan containing axis Oz is

$$u_s^0 = \left[\left(1 - \frac{2mr}{\sum}\right) - \frac{2\omega}{\sum} \cdot 2mar \sin^2\theta - \omega^2 \left(r^2 + a^2 \frac{2ma^2 r \sin\theta}{\sum}\right) \sin^2\theta\right]^{-\frac{1}{2}}. \tag{15}$$

At the poles where $\theta = 0$ or $\theta = \pi$ we get

$$u_s^0 = \left(1 - \frac{2mr_p}{r_p^2 + a^2}\right)^{-\frac{1}{2}} \tag{16}$$

and at the equator where $\theta = \dfrac{\pi}{2}$, we get

$$u_s^0 = \left[\left(1 - \frac{2m}{r_e}\right) - 2\omega a \frac{2m}{r_e} - \omega^2 \left(r_e^2 + a^2 + a^2 \frac{2m}{r_e}\right)\right]^{-\frac{1}{2}}. \tag{17}$$

Equalizing u_s^0 from (15) to the one from (16) we get for the surface of the body the equation:

$$\frac{2mr}{\sum} + 2\omega \frac{2ar}{\sum} \sin^2\theta + \omega^2 \left(r^2 + a^2 + \frac{2ma^2 r \sin\theta}{\sum}\right) \sin^2\theta = \frac{2mr_p}{r_p^2 + a^2} \tag{18}$$

In order to able to study the shape of the curve (18) we express the generalized magnitudes compared to the polar radius r_p of the body introducing the notations:

$$w = \frac{r}{r_p}; \quad \beta = \frac{r_e}{r_p}; \quad \gamma = \omega r_p; \quad \mu = \frac{2m}{r_p}; \quad \sigma = \frac{a}{r_p} \tag{19}$$

Equation (18) becomes

$$\frac{\mu w + 2\gamma\mu\sigma w \sin^2\theta}{w^2 + \sigma^2\cos^2\theta} + \gamma^2\left(w^2 + \sigma^2 + \frac{\mu\sigma^2 w \sin^2\theta}{w^2 + \sigma^2\cos^2\theta}\right)\sin^2\theta = \frac{\mu}{1+\sigma^2} \tag{20}$$

In the case of the Earth:

$$\gamma = 1.546\cdot 10^{-6}; \quad \sigma = 5.142\cdot 10^{-6}; \quad \sigma^2 = 2.64\cdot 10^{-11}; \quad \mu = 1.880\cdot 10^{-9} \tag{21}$$

σ^2 and $\gamma\sigma$ being very little as compared to w^2 can be neglected and from (20) results

$$\frac{w^3\sin^2\theta}{w-1} = \frac{\mu}{\gamma^2} = k \tag{22}$$

The equation (22) is the shape-equation for Earth and for the Earth-type planets. In physical values:

$$k = \frac{\mu}{\gamma^2} = \frac{G}{2\pi^2}\cdot\frac{M_{ph}T^2}{r_p^3} \tag{23}$$

$T =$ planets sidereal day (rotational period).

In case of Earth [2], [4] $M_{ph} = 5.9736\cdot 10^{24}kg$; $T = 23^h 56^m 04^s \cdot 1 = 86164.1s$; $r_p = 6.3568\cdot 10^6 m$, $r_e = 6.3781\cdot 10^6 m$.

For the equator $\theta = \frac{\pi}{2}$, $w = \beta$.

$$k = \frac{\beta^3}{\beta - 1} \quad \text{and} \quad T^2 = \frac{2\pi^2}{G}\cdot k\cdot\frac{r_p^3}{M_{ph}}$$

In the Earth case $\beta = 1.003351$ and $k = 301.4513341$ corresponds to the calculated value $T = 61922.2s$

Taking into account that, as it results from the present measurements, each day diminishes with approximately 0.001 seconds per century, it results that our calculated period matches the one about $2.4\cdot 10^9$ years ago when the terrestrial crust coagulated. For the planet Mars

$$r_e = 3.397\cdot 10^6 m; \quad r_p = 3.375\cdot 10^6 m; \quad \beta = 1.006518; \quad k = 156.4287$$

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