CREATIVE MATH. & INF.
17 (2008), 127 - 136

A note on the Nagel and Gergonne points

DORIN ANDRICA AND KHOA LU NGUYEN

ABSTRACT. In this note, we study some special properties of the Nagel point, Gergonne point, and their adjoints.

1. INTRODUCTION

Given a triangle ABC, denote by O the circumcenter, Ge the Gergonne point, and N the Nagel point of ABC. It is known that the distance ON is equal to $R-2r$, here R and r are respectively the circumradius and inradius of ABC (see, for instance, [1]). In Section 4, we will see that the first part of the famous Feurbach's theorem, which states that the nine-point circle is intouched by the incircle, follows from this beautiful distance.

What about the second part of Feurbach's theorem, which states that the nine-point circle is extouched by the excircles? With similar direction, we shall use the points N_a, N_b, N_c, as the adjoints of the Nagel point N, that share similar properties of N. The adjoint Nagel points N_a, N_b, N_c are extensively studied in [5, pp. 260-293]. Also, in this paper, we will study some interesting properties of this configuration together with the adjoints G_a, G_b, G_c of the Gergonne point Ge which are defined as the isotomic conjugate of N_a, N_b, N_c respectively.

2. THE NAGEL CONFIGURATION

Let G, H, I are respectively the centroid, orthocenter, and incenter of ABC. We first study an important characteristic of the Nagel point N.

Theorem 2.1. *The Furhmann center F of ABC is the midpoint of HN; and the centroid G divides the segment NI with ratio -2.*

Received: 11.01.2008. In revised form: 24.01.2008.
2000 *Mathematics Subject Classification.* 51A20, 51M04.
Key words and phrases. *Nagel point, Gergonne point, Nagel adjoint points, Gergonne adjoints points, Feurbach's theorem, complex coordinates.*

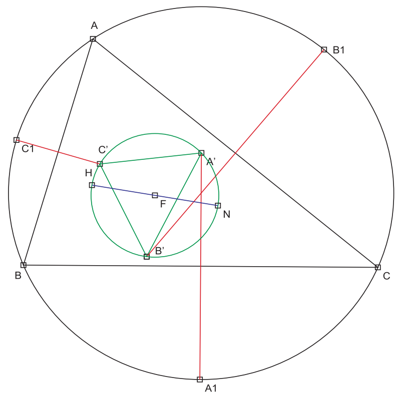

Figure 1

Proof. Let A_1, B_1, C_1 be the midpoints of the arcs BC, CA, AB not containing the points A, B, C respectively. Suppose A', B', C' are the reflections of A_1, B_1, C_1 through their corresponding sides BC, CA, AB. The Furhmann center is defined as the circumcenter of $A'B'C'$. In order to prove F is the midpoint of HN, we shall prove that HN is a diameter of the circumcircle of $A'B'C'$, i.e. the Furhmann circle.

Consider the reflection through the midpoint M of BC that maps the orthocenter H to the antipode U of A, the point A' to A_1, the Nagel point to a point T_a, and the incenter I to a point J_a. Since U is the antipode of A, it is sufficient to prove that T_a lies on the line AA_1.

Suppose A_a is the point of tangency of the A-excircle (I_a) with BC. It follows that A, N, A_a are collinear and I_a, J_a, A_a are collinear. We have

$$\frac{A_aJ_a}{J_aI_a} = \frac{r}{r_a} = \frac{s-a}{s}\ ; \quad \frac{AI}{AI_a} = \frac{r}{r_a} = \frac{s-a}{s}$$

$$\frac{NK}{AK} = \frac{1}{1+\dfrac{AN}{NK}} = \frac{1}{1+\dfrac{s-b}{s-a}+\dfrac{s-b}{s-a}} = \frac{s-a}{s}$$

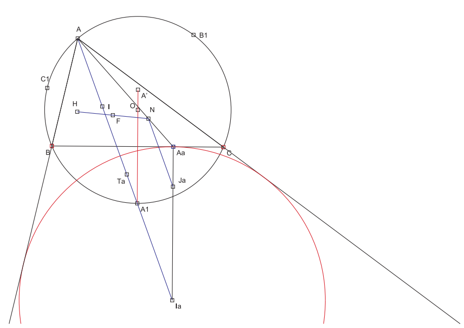

Figure 2

Thus,

$$\frac{A_aJ_a}{J_aI_a} = \frac{AI}{AI_a} = \frac{NK}{AK}.$$

Hence, we obtain NJ_a is parallel to AI_a, and IJ_a is parallel to AA_a.

On the other hand, since INJ_aT_a is a parallelogram, we obtain NJ_a is parallel to IT_a. Since AI_a and IT_a both pass through the incenter I and are parallel to IT_a, it follows that T_a lies on the bisector AI_a. Since A_1 lies on AI_a, we obtain T_a lies on AA_1. Therefore, F is the midpoint of HN.

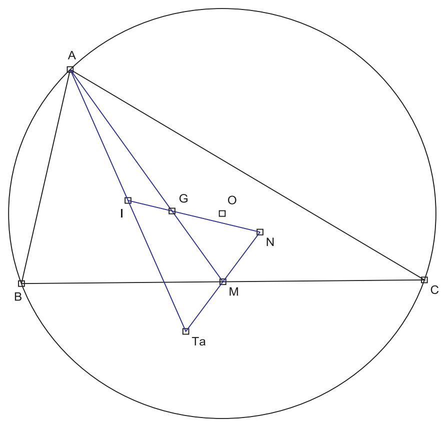

Figure 3

Since M is the midpoint of T_aN and MI is parallel to AN, we infer that I is the midpoint of AT_a. Applying Menelaus's theorem to triangle AT_aM and three points I, G, N it follows that I, G, N are collinear and $\dfrac{GN}{GI} = -2$. The proof is complete. $\square$

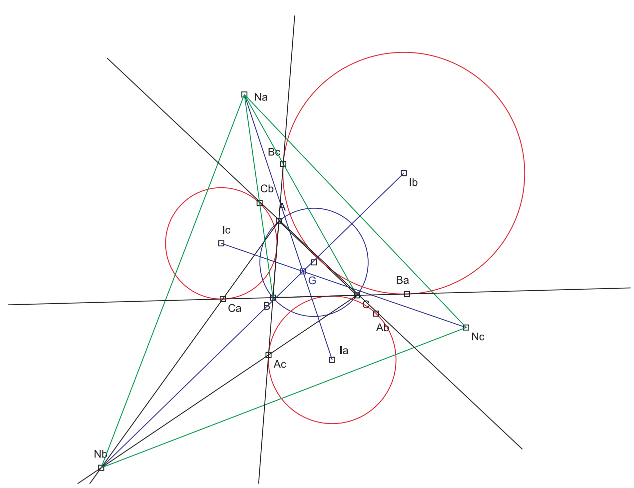

Figure 4

Let A_b and A_c be the points of tangency of the A-excircle with AC and AB. Define B_c and B_a, C_a and C_b respectively. Let N_a, N_b, N_c be respectively the intersections of BB_a and CC_a, CC_b and AA_b, AA_c and BB_c. Call A_2, B_2, C_2 the midpoints of the arcs BC, CA, AB containing A, B, C and suppose $A'2$, $B'2$, $C'2$ be their reflections to the corresponding sides BC, CA, AB. Denote by F_a, F_b, F_c the circumcenters of triangles $A_1'B_2'C_2'$, $B_1'C_2'A_2'$, $C_1'A_2'B_2'$. Then by Conway's extraversion to Theorem 2.1, it follows that F_a, F_b, F_c are respectively the midpoints of HN_a, HN_b, HN_c and the centroid G divides the segments I_aN_a, I_bN_b, I_cN_c with ratio -2. Thus, we get the following result.

Theorem 2.2. *The triangle $N_aN_bN_c$ is homothetic to the external triangle $I_aI_bI_c$ with center G and ratio -2.*

Since I is the orthocenter of $I_aI_bI_c$, we obtain the following consequence.

Corollary 2.1. *The Nagel point N is the orthocenter of $N_aN_bN_c$.*

Suppose UVW is the antimedial triangle of ABC. Since

$$\frac{GU}{GA} = \frac{GV}{GB} = \frac{GW}{GC} = -2$$

and ABC is the orthic triangle of $I_aI_bI_c$, another corollary follows.

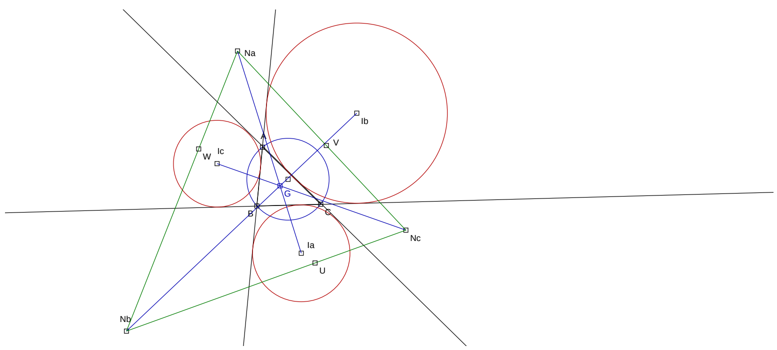

Figure 5

Corollary 2.2. *The antimedial triangle UVW of ABC is the orthic triangle of $I_aI_bI_c$.*

We end this section by a simple theorem inferred from Ceva's theorem.

Theorem 2.3. *Triangles $N_aN_bN_c$ and ABC are perspective at the Gergonne point Ge.*

3. The triangle $G_aG_bG_c$

We first state a simple result following from the definition and Ceva's theorem.

Theorem 3.1. *Triangle $G_aG_bG_c$ and ABC are perspective at the Nagel point N.*

In this section, we give an interesting synthetic proof to the following property.

Theorem 3.2. *The triangle $G_aG_bG_c$ and the excentral triangle $I_aI_bI_c$ are perspective at the de Longchamp point L.*

In order to prove this theorem, we need the following lemmas.

Lemma 3.1. *The four lines I_bI_c, G_bG_c, B_bC_c and B_cC_b are concurrent.*

Proof. We first notice that C_c and B_b are reflections of C_b and B_c through I_bI_c respectively. Hence B_bC_c is the reflection of B_cC_b through I_bI_c. It follows that B_bC_c, B_cC_b, I_bI_c are concurrent. It is sufficient now to prove that B_bC_c, B_cC_b and G_bG_c are concurrent. Consider two triangles $B_cG_bB_b$ and $C_bG_cC_c$. Notice that N_a is the intersection of G_bB_c and G_cC_b, N is the intersection of G_bB_b and G_cC_c. If we can show that NN_a, B_bB_c and C_cC_b are concurrent, then by Desargues' theorem, we are done.

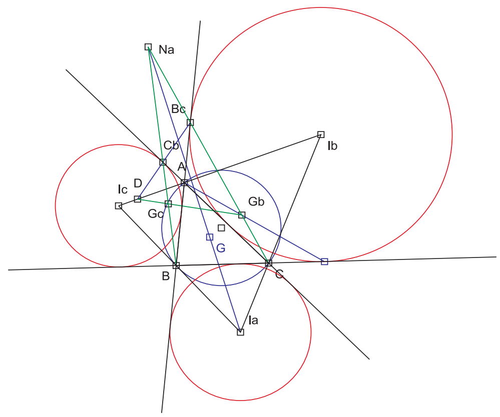

Figure 6

From Section 2, we know that NN_a is parallel to II_a. Since II_a is perpendicular to I_bI_c, NN_a is perpendicular to I_bI_c. On the other hand, since C_c and B_b are reflections of C_b and B_c through I_bI_c respectively, it follows that B_bB_c and C_cC_b are both perpendicular to I_bI_c. Thus, we obtain NN_a, B_bB_c, C_cC_b are concurrent at a point at infinity. Therefore, the lemma is completely proved. □

Lemma 3.2. *The circumcenter O' of $I_aI_bI_c$ is the reflection of I through O.*

It is also well known that O' is the perspector of triangle $I_aI_bI_c$ and $A_aB_bC_c$ since I_aO' is perpendicular to BC, I_bO' is perpendicular to CA, I_cO' is perpendicular to AB.

Lemma 3.3. *Let X, Y, Z be the intersections of I_cC_b and I_bB_c, I_cC_a and I_aA_c, I_aA_b and I_bB_a. Then $N_aN_bN_c$ and XYZ are homothetic with ratio 2.*

Proof. Since I_aO' is perpendicular to BC, I_bO' is perpendicular to CA, I_cO' is perpendicular to AB, we obtain I_cYI_aO' and I_bZI_aO' are parallelograms. Thus, I_bI_cYZ is also a parallelogram. This yields that $I_bI_c = YZ$ and they are parallel in opposite directions. By similar argument, we obtain that $I_aI_bI_c$ are homothetic with ratio -1. From Section 1, we know that triangles $N_aN_bN_c$ and $I_aI_bI_c$ are homothetic with ratio -2. Therefore, $N_aN_bN_c$ and XYZ are homothetic with ratio 2. The lemma is proved. □

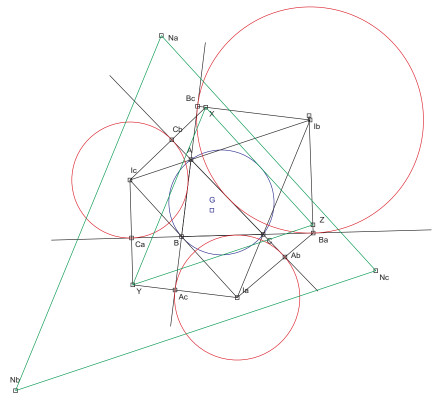

Figure 7

We will now prove Theorem 3.2.

Proof. We first show that $I_aI_bI_c$ and $G_aG_bG_c$ are perspective at a point L. From Lemma 3.2, we have I_bI_c, G_bG_c, B_bC_c are concurrent. Denote this point D. Similarly, we define the points E and F.

Since triangle $I_aI_bI_c$ is perspective with $A_aB_bC_c$ at the point O', this implies from Desargues's theorem that D, E, F are collinear. Again, since D, E, F are collinear, by Desargues's theorem, we obtain $I_aI_bI_c$ and $G_aG_bG_c$ are perspective at a point L.

Now, we shall prove that L is the De Longchamp point. We shall need the Lemma 3.2 again. We know that I_bI_c, B_cC_b, G_bG_c are concurrent. Consider two perspective triangles $I_bG_bB_c$ and $I_cG_cC_b$. Then by Desargues' theorem, we obtain that L lies on N_aX. By similar arguments, it follows that N_aX, N_bY, N_cZ are concurrent at L.

From Lemma 3.3, we know that $N_aN_bN_c$ and XYZ are homothetic with ratio 2. Hence L is the center of similitude. Since N and O' are respectively the orthocenter of $N_aN_bN_c$ and XYZ, we obtain that $\frac{LN}{LO'} = 2$. Since

$$\frac{GI}{GN}\frac{LN}{LO'}\frac{OO'}{OI} = 1,$$

according to Menelaus' theorem, three points G, O, L are collinear. Applying Menelaus's theorem again to triangle NGL and three collinear points I, O, O', we obtain that $\frac{OL}{OG} = -3$. By definition, it yields that L is the de Longchamp point of ABC. And the theorem is proved. □

4. Complex coordinates approach

In this section, we shall compute the distance between the circumcircle O and N_a by using complex numbers.

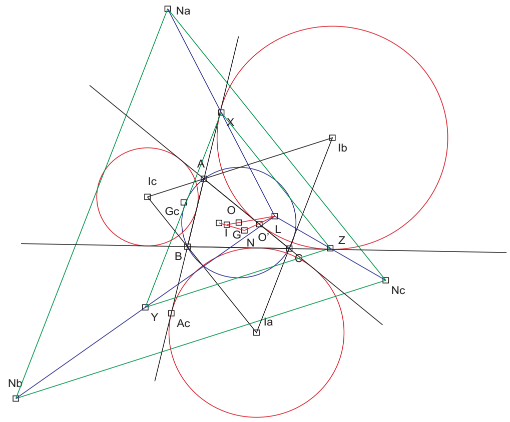

Figure 8

Assume that the circumcenter O of triangle ABC is the origin of the complex plan and let z_a, z_b, z_c be the complex coordinates of A, B, C respectively. It is not difficult to calculate the complex coordinates of the points G, H, I, N, I_a, I_b, I_c (see [1, pp. 103-105]):

$$z_G = \frac{z_A + z_B + z_C}{3}, z_H = z_A + z_B + z_C, z_I = \frac{az_A + bz_B + cz_C}{2s}$$

$$z_N = \frac{(s-a)z_A + (s-b)z_B + (s-c)z_C}{s}, z_{I_a} = \frac{-az_A + bz_B + cz_C}{2(s-a)}$$

These coordinates are used in the book [1] to solve various geometric problems, for instance to determine some important distances in triangle ABC. The distance $ON = R - 2r$ has already been mentioned in the introduction (see [1, Subsection 4.6.3, Theorem 6]). This beautiful relation is equivalent to the first part of Feurbach's theorem, which states that the nine-point circle is intouched by the incircle. Indeed, we have the following configuration, where points O, G, W, H lies on the Euler line while I, G, N lie on the Nagel line.

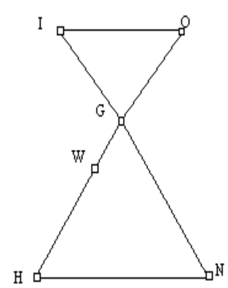

Figure 9

Since $\dfrac{GI}{GN}=\dfrac{GW}{GO}=\dfrac{1}{2}$, we obtain IW is parallel to ON and $IW=\dfrac{ON}{2}=\dfrac{R}{2}-r$. Notice that $\dfrac{R}{2}$ is the radius of the nine-point circle, r is the radius of the incircle and IW is the distance between the two centers. Therefore, the nine-point circle is intouched by the incircle.

We will now proceed to calculate the distance ON_a. From Section 2, we know that $\dfrac{GN_a}{GI_a}=-2$. Thus, $z_G-z_{N_a}=2(z_{I_a}-z_G)$. Using the complex coordinates of G and I_a above, it follows that

$$z_{N_a}=\frac{sz_A-(s-c)z_B-(s-b)z_C}{s-a}=\frac{r_a}{r}z_A-\frac{r_a}{r_c}z_B-\frac{r_a}{r_b}z_C$$

We shall calculate the distance ON_a by using the real product of two complex number z_1 and z_2 and defined as

$$z_1\cdot z_2=\frac{z_1\overline{z}_2+\overline{z}_1z_2}{2}.$$

It is not difficult to see that the real product is a real number, it is commutative, distributive with respect to addition and satisfies

$$z\cdot z=|z|^2.$$

We also have

$$z_A\cdot z_B=R^2-\frac{c^2}{2},\quad z_B\cdot z_C=R^2-\frac{a^2}{2},\quad z_C\cdot z_A=R^2-\frac{b^2}{2}.$$

Indeed, it follows from

$$c^2=|z_A-z_B|^2=(z_A-z_B)\cdot(z_A-z_B)=z_A^2-2z_A\cdot z_B+z_B^2=2R^2-2z_Az_B.$$

Thus

$$\begin{aligned}ON_a^2=|z_{N_a}|^2&=r_a^2R^2\left(\frac{1}{r^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)\\&+r_a^2\left(\frac{2}{r_br_c}z_B\cdot z_C-\frac{2}{rr_c}z_A\cdot z_B-\frac{2}{rr_b}z_A\cdot z_C\right)\\&=r_a^2R^2\left(\frac{1}{r^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)\\&+r_a^2\left[\frac{2}{r_br_c}\left(R^2-\frac{a^2}{2}\right)-\frac{2}{rr_c}\left(R^2-\frac{c^2}{2}\right)-\frac{2}{rr_b}\left(R^2-\frac{b^2}{2}\right)\right]\\&=r_a^2R^2\left(-\frac{1}{r}+\frac{1}{r_b}+\frac{1}{r_c}\right)^2\\&+r_a^2\left(\frac{-a^2}{r_br_c}+\frac{b^2}{rr_b}+\frac{c^2}{rr_c}\right)\\&=R^2+r_a^2\left(\frac{-a^2}{r_br_c}+\frac{b^2}{rr_b}+\frac{c^2}{rr_c}\right).\end{aligned}$$

Now, let us prove that the following relation

$$\frac{-a^2}{r_br_c}+\frac{b^2}{rr_b}+\frac{c^2}{rr_c}=\frac{4R}{r_a}+4.$$

By replacing

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, \quad r_a = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

etc., we obtain

$$\begin{aligned}
&\frac{-a^2}{r_br_c}+\frac{b^2}{rr_b}+\frac{c^2}{rr_c}\\
&=\frac{4}{\sin A\sin B\sin C}\left(-\sin^2\frac{A}{2}\sin A+\cos^2\frac{B}{2}\sin B+\cos^2\frac{C}{2}\sin C\right)\\
&=\frac{2}{\sin A\sin B\sin C}[-(1-\cos A)\sin A+(1+\cos B)\sin B+(1+\cos C)\sin C]\\
&=\frac{2(\sin B+\sin C-\sin A)}{\sin A\sin B\sin C}+\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A\sin B\sin C}\\
&=\frac{8\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin A\sin B\sin C}+\frac{\sin A\sin B\sin C}{\sin A\sin B\sin C}\\
&=\frac{1}{\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}+4=\frac{4R}{r_a}+4.
\end{aligned}$$

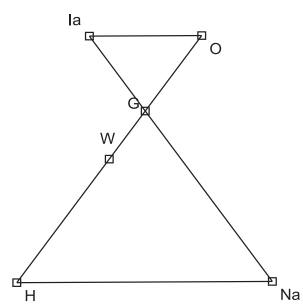

Figure 10

Therefore, we obtain

$$ON_a^2 = R^2 + r_a^2\left(\frac{4R}{r_a}+4r_a^2\right) = (R+2r_a)^2 \quad \text{or} \quad ON_a = R + 2r_a.$$

The last relation also appears in [5, pp. 283], where it derived from the Feurbach's theorem. We have obtained a direct proof for the distance ON_a.

The above relation is equivalent to the second part of Feurbach's theorem. Consider the following configuration, where I_a, G, N_a is another line of Nagel type. We have

$$I_aW = \frac{ON_a}{2} = \frac{R}{2} + r_a.$$

Therefore, the A-excircle (I_a) extouches the nine-point circle. Similarly, the other excircles also extouch the nine-point circle, and the Feurbach's second part is completely proved.

REFERENCES

[1] Andreescu, T., Andrica, D., *Complex Number from A to... Z*, Birkhauser, Boston-Basel-Berlin, 2006
[2] Coxeter, H.S.M, Greitzer, S.L.,*Geometry Revisited*, Random House, New York, 1967
[3] Honsberger, R., *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, The Mathematical Association of America, New Mathematical Library 37, 1995
[4] Johnson, R.A., *Advanced Euclidean Geometry*, New York, 1960
[5] Mihălescu, C., *The Geometry of Remarkable Elements. Some Points, Lines and Circles* (Romanian), Editura Tehnică, Bucharest, 1957

DORIN ANDRICA
BABEŞ-BOLYAI UNIVERSITY
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
40084 CLUJ-NAPOCA, ROMANIA
E-mail address: dandrica@math.ubbcluj.ro

KHOA LU NGUYEN
MASSACHUSETTS INSTITUTE OF TECHNOLOGY, STUDENT
77 MASSACHUSETTS AVENUE, CAMBRIDGE, MA, 02139-4307
E-mail address: treegoner@yahoo.com