CREATIVE MATH. & INF.
17 (2008), 137 - 141

A category of new inequalities

MIHÁLY BENCZE

ABSTRACT. In this paper we present a category of new inequalities generated by weighted AM - GM - HM inequalities.

1. MAIN RESULTS

Theorem 1.1. *If* $a_k, x, y, z > 0$ $(k = 1, 2, ..., n)$ *and* $i \in \{1, 2, ..., n\}$, *then*

$$\left(\frac{a_1}{a_2}\right)^x + \left(\frac{a_2}{a_3}\right)^x + ... + \left(\frac{a_n}{a_1}\right)^x \geq$$
$$\geq \left(\frac{a_1}{a_{i+1}}\right)^{\frac{x(y-z)}{iy+(n-i)z}} + \left(\frac{a_2}{a_{i+2}}\right)^{\frac{x(y-z)}{iy+(n-i)z}} + ... + \left(\frac{a_n}{a_i}\right)^{\frac{x(y-z)}{iy+(n-i)z}}.$$

Proof. Using the weighted AM-GM-HM inequality we have

$$y\left(\frac{a_1}{a_2}\right)^x + ... + y\left(\frac{a_i}{a_{i+1}}\right)^x + z\left(\frac{a_{i+1}}{a_{i+2}}\right)^x + ... + z\left(\frac{a_n}{a_1}\right)^x \geq$$
$$\geq (iy + (n-i)z)\left(\left(\frac{a_1}{a_2}...\frac{a_i}{a_{i+1}}\right)^y \left(\frac{a_{i+1}}{a_{i+2}}...\frac{a_n}{a_1}\right)^z\right)^{\frac{x}{iy+(n-i)z}} =$$
$$= (iy + (n-i)z)\left(\frac{a_1}{a_{i+1}}\right)^{\frac{x(y-z)}{iy+(n-i)z}},$$

therefore $(iy + (n-i)z)\left(\frac{a_1}{a_2}\right)^x =$

$$= \sum_{cyclic}\left(y\left(\frac{a_1}{a_2}\right)^x + ... + y\left(\frac{a_i}{a_{i+1}}\right)^x + z\left(\frac{a_{i+1}}{a_{i+2}}\right)^x + ... + z\left(\frac{a_n}{a_1}\right)^x\right) \geq$$
$$\geq (iy + (n-i)z)\sum_{cyclic}\left(\frac{a_1}{a_{i+1}}\right)^{\frac{x(y-z)}{iy+(n-i)z}}.$$

□

Corollary 1.1. *The sequence* $(x_k)_{k\geq 0}$, *where* $x_k = \left(\frac{a_1}{a_2}\right)^k + \left(\frac{a_2}{a_3}\right)^k + ... + \left(\frac{a_n}{a_1}\right)^k$, *is increasing and* $... \geq x_{k+1} \geq x_k \geq ... \geq x_0 = n$.

Theorem 1.2. *If* $x, a_k, p_k > 0$ $(k = 1, 2, ..., n)$ *then*

$$\sum_{cyclic}\left(\frac{a_1}{a_2}\right)^x \geq \sum_{cyclic}\left(a_1^{-(p_n-p_1)}a_2^{p_2-p_1}...a_n^{p_n-p_{n-1}}\right)^{\frac{x}{\sum_{k=1}^{n}p_k}}.$$

Received: 29.10.2007. In revised form: 5.11.2007.
2000 *Mathematics Subject Classification.* 26D15.
Key words and phrases. *Weighted arithmetical, geometrical, harmonical means etc.*

Proof. Using the weighted AM-GM inequality we have:

$$p_1\left(\frac{a_1}{a_2}\right)^x+p_2\left(\frac{a_2}{a_3}\right)^x+...+p_n\left(\frac{a_n}{a_1}\right)^x\geq$$
$$\geq(p_1+...+p_n)\left(\left(\frac{a_1}{a_2}\right)^{p_1}\left(\frac{a_2}{a_3}\right)^{p_2}...\left(\frac{a_n}{a_1}\right)^{p_n}\right)^{\frac{x}{\sum\limits_{k=1}^n p_k}}=$$
$$=\left(\sum_{k=1}^n p_k\right)\left(a_1^{-(p_n-p_1)}a_2^{p_2-p_1}...a_n^{p_n-p_{n-1}}\right)^{\frac{x}{\sum\limits_{k=1}^n p_k}},$$

therefore $\left(\sum\limits_{k=1}^n p_k\right)\sum\limits_{cyclic}\left(\frac{a_1}{a_2}\right)^x=\sum\limits_{cyclic}\left(p_1\left(\frac{a_1}{a_2}\right)^x+p_2\left(\frac{a_2}{a_3}\right)^x+...+\right.$

$\left.+...+p_n\left(\frac{a_n}{a_1}\right)^x\right)\geq\left(\sum\limits_{k=1}^n p_k\right)\sum\limits_{cyclic}\left(a_1^{-(p_n-p_1)}a_2^{p_2-p_1}...a_n^{p_n-p_{n-1}}\right)^{\frac{x}{\sum\limits_{k=1}^n p_k}}.$ □

Theorem 1.3. *If $a_k,x,y,z,t>0$ $(k=1,2,...,n)$ and $i\in\{1,2,...,n\}$, then*

$$a_1^xa_2^y+a_2^xa_3^y+...+a_n^xa_1^y\geq\sum_{cyclic}a_1^{\frac{xz+yt}{ix+(n-i)t}}a_{i+1}^{\frac{yz+xt}{iz+(n-i)t}}\left(a_2...a_ia_{i+2}...a_n\right)^{\frac{(x+y)t}{iz+(n-i)t}}.$$

Proof. Using the weighted AM-GM inequality we have:

$$za_1^xa_2^y+...+za_i^xa_{i+1}^y+ta_{i+1}^xa_{i+2}^y+...+ta_n^xa_1^y\geq$$
$$\geq(iz+(n-i)t)\left(a_1^xa_2^y...a_i^xa_{i+1}^y\right)^{\frac{z}{iz+(n-i)t}}\cdot\left(a_{i+1}^xa_{i+2}^y...a_n^xa_1^y\right)^{\frac{t}{iz+(n-i)t}}=$$
$$=(iz+(n-i)t)\,a_1^{\frac{xz+yt}{iz+(n-i)t}}a_{i+1}^{\frac{yz+xt}{iz+(n-i)t}}\left(a_2...a_ia_{i+2}...a_n\right)^{\frac{(x+y)t}{iz+(n-i)t}},$$

therefore

$$(iz+(n-i)t)\sum_{cyclic}a_1^xa_2^y=$$
$$=\sum_{cyclic}\left(za_1^xa_2^y+...+za_i^xa_{i+1}^y+ta_{i+1}^xa_{i+2}^y+...+ta_n^xa_1^y\right)\geq$$
$$\geq(iz+(n-i)t)\sum_{cyclic}a_1^{\frac{xz+yt}{iz+(n-i)t}}a_{i+1}^{\frac{yz+xt}{iz+(n-i)t}}\left(a_2...a_ia_{i+2}...a_n\right)^{\frac{(x+y)t}{iz+(n-i)t}}.$$

□

Theorem 1.4. *If $a_k,p_k,x,y>0$ $(k=1,2,...,n)$ then*

$$\sum_{cyclic}a_1^xa_2^y\geq\sum_{cyclic}\left(a_1^{p_1x+p_ny}a_2^{p_1y+p_2x}...a_n^{p_{n-1}y+p_nx}\right)^{\frac{1}{\sum\limits_{k=1}^n p_k}}.$$

Proof. Using the weighted AM-GM inequality we obtain:

$$p_1a_1^xa_2^y+p_2a_2^xa_3^y+...+p_na_n^xa_1^y\geq$$
$$\geq\left(\sum_{k=1}^n p_k\right)\left(\left(a_1^xa_2^y\right)^{p_1}\left(a_2^xa_3^y\right)^{p_2}...\left(a_n^xa_1^y\right)^{p_n}\right)^{\frac{1}{\sum\limits_{k=1}^n p_k}}=$$
$$=\left(\sum_{k=1}^n p_k\right)\left(a_1^{p_1x+p_nx}a_2^{p_1y+p_2x}...a_n^{p_{n-1}y+p_nx}\right)^{\frac{1}{\sum\limits_{k=1}^n p_k}},$$

therefore

$$\left(\sum_{k=1}^{n} p_k\right) \sum_{cyclic} a_1^x a_2^y = \sum_{cyclic} \left(p_1 a_1^x a_2^y + ... + p_n a_n^x a_1^y\right) \geq$$
$$\geq \left(\sum_{k=1}^{n} p_k\right) \sum_{cyclic} \left(a_1^{p_1 x + p_n y} a_2^{p_1 y + p_2 x} ... a_n^{p_{n-1} y + p_n x}\right)^{\frac{1}{\sum_{k=1}^{n} p_k}}.$$

□

Theorem 1.5. *If* $a_k, x, y, p_k > 0$ $(k = 1, 2, ..., n)$, *then*

$$\sum_{cyclic} \frac{a_1^x}{a_2^y} \geq \sum_{cyclic} \left(a_1^{p_1 x - p_n y} a_2^{p_2 x - p_1 y} ... a_n^{p_n x - p_{n-1} y}\right)^{\frac{1}{\sum_{k=1}^{n} p_k}}.$$

Proof. Using the weighted AM-GM inequality we have:

$$p_1 \frac{a_1^x}{a_2^y} + p_2 \frac{a_2^x}{a_3^y} + ... + p_n \frac{a_n^x}{a_1^y} \geq \left(\sum_{k=1}^{n} p_k\right) \left(\left(\frac{a_1^x}{a_2^y}\right)^{p_1} ... \left(\frac{a_n^x}{a_1^y}\right)^{p_n}\right)^{\frac{1}{\sum_{k=1}^{n} p_k}} =$$
$$= \left(\sum_{k=1}^{n} p_k\right) \left(a_1^{p_1 x - p_n y} a_2^{p_2 x - p_1 y} ... a_n^{p_n x - p_{n-1} y}\right)^{\frac{1}{\sum_{k=1}^{n} p_k}}$$

After addition we obtain the desired inequality. □

Theorem 1.6. *If* $a_k, p_k, x, y, z > 0$ $(k = 1, 2, ..., n)$, *then*

$$\sum_{cyclic} \frac{a_1^x}{a_2^y a_3^z} \geq \sum_{cyclic} \left(a_1^{p_1 x - p_{n-1} y - p_n z} ... a_n^{p_n x - p_{n-2} y - p_n z}\right)^{\frac{1}{\sum_{k=1}^{n} p_k}}.$$

Proof. Using the weighted AM-GM inequality we have:

$$p_1 \frac{a_1^x}{a_2^y a_3^z} + p_2 \frac{a_2^x}{a_3^y a_4^z} + ... + p_n \frac{a_n^x}{a_1^y a_2^z} \geq \left(\sum_{k=1}^{n} p_k\right) \left(\left(\frac{a_1^x}{a_2^y a_3^z}\right)^{p_1} ... \left(\frac{a_n^x}{a_1^y a_2^z}\right)^{p_n}\right)^{\frac{1}{\sum_{k=1}^{n} p_k}} =$$
$$= \left(\sum_{k=1}^{n} p_k\right) \left(a_1^{p_1 x - p_{n-1} y - p_n z} a_2^{p_2 x - p_1 y - p_n z} ... a_3^{p_3 x - p_2 y - p_1 z} ... a_n^{p_n x - p_{n-1} y - p_{n-2} z}\right)^{\frac{1}{\sum_{k=1}^{n} p_k}}.$$

After addition yields the desired inequality. □

Theorem 1.7. *If* $\alpha, a_k > 0$ *and* $p_k \geq 1$ $(k = 1, 2, ..., n)$, *then*

$$\frac{1}{\alpha}\left(\left(\frac{a_1}{a_2}\right)^{\alpha} + \left(\frac{a_2}{a_3}\right)^{\alpha} + ... + \left(\frac{a_n}{a_1}\right)^{\alpha}\right) + \sum_{k=1}^{n} p_k \geq$$

$$\geq \frac{n}{\alpha} + p_1 \left(\frac{a_1}{a_2}\right)^{\frac{1}{p_1}} + p_2 \left(\frac{a_2}{a_3}\right)^{\frac{1}{p_2}} + ... + p_n \left(\frac{a_n}{a_1}\right)^{\frac{1}{p_n}}$$

Proof. We have $\frac{1}{\alpha}\left(\frac{a_k}{a_{k+1}}\right)^{\alpha} + \frac{p_k \alpha - 1}{\alpha} \geq p_k \left(\frac{a_k}{a_{k+1}}\right)^{\frac{1}{p_k}}$ $(k = 1, 2, ..., n)$. After summation we obtain the desired inequality. □

Theorem 1.8. *If* $a_i > 0\ (i = 1, 2, ..., n)\,, p_1, p_2, ..., p_k > 0,\ k \in \{1, 2, ..., n\}\,,$ *then*

$$\frac{1}{p_1 + p_2 + ... + p_k} \sum_{i=1}^{n} \frac{1}{a_i} \geq$$
$$\geq \frac{1}{p_1 a_1 + p_2 a_2 + ... + p_k a_k} + \frac{1}{p_1 a_2 + p_2 a_3 + ... + p_k a_{k+1}} + ...+$$
$$+... + \frac{1}{p_1 a_n + p_2 a_1 + ... + p_k a_{k-1}} \geq \frac{n^2}{(p_1 + p_2 + ... + p_k) \sum\limits_{i=1}^{n} a_i}.$$

Proof. We have $(p_1 + p_2 + ... + p_k) \sum\limits_{i=1}^{n} \frac{1}{a_i} = \sum\limits_{cyclic} \left(\frac{p_1}{a_1} + \frac{p_2}{a_2} + ... + \frac{p_k}{a_k} \right) \geq$
$\geq \sum\limits_{cyclic} \frac{(p_1 + ... + p_k)^2}{p_1 a_1 + p_2 a_2 + ... + p_k a_k}$ etc. □

Theorem 1.9. *If* $x_1, x_2, ..., x_k, a_i, p_i > 0\ (k = 1, 2, ..., n)\,, k \in \{1, 2, ..., n\}\,,$ *then*

$$\frac{1}{x_1 a_1 + x_2 a_2 + ... + x_k a_k} + \frac{1}{x_1 a_2 + x_2 a_3 + ... + x_k a_{k+1}} + ...+$$
$$+... + \frac{1}{x_1 a_n + x_2 a_1 + ... + x_k a_{k-1}} \geq \left(\sum_{i=1}^{n} p_i \right)$$
$$\left(\frac{1}{q_1 a_1 + q_2 a_2 + ... + q_n a_n} + \frac{1}{q_1 a_2 + q_2 a_3 + ... + q_n a_1} + ...+ \right.$$
$$\left. + \frac{1}{q_1 a_n + q_2 a_1 + ... + q_n a_{n-1}} \right) \geq \frac{n^2}{(x_1 + x_2 + ... + x_k) \sum\limits_{i=1}^{n} a_i}$$

where $q_1 = p_1 x_1 + p_n x_2 + ... + p_{n-k+2} x_k;$
$q_2 = p_2 x_1 + p_1 x_2 + ... + p_{n-k+3} x_k; ...; q_n = p_n x_1 + p_{n-1} x_2 + ... + p_{n-k+1} x_k.$

Proof. We have

$$\frac{p_1}{x_1 a_1 + x_2 a_2 + ... + x_k a_k} + \frac{p_2}{x_1 a_2 + x_2 a_3 + ... + x_k a_{k+1}} + ...+$$
$$+... + \frac{p_n}{x_1 a_n + x_2 a_1 + ... + x_k a_{k-1}} \geq$$
$$\geq \frac{\left(\sum\limits_{i=1}^{n} p_i \right)^2}{\sum\limits_{cyclic} (x_1 a_1 + x_2 a_2 + ... + x_k a_k)} = \frac{\left(\sum\limits_{i=1}^{n} p_i \right)^2}{q_1 a_1 + q_2 a_2 + ... + q_n a_n},$$

therefore

$$\left(\sum_{i=1}^{n} p_i\right)^2 \sum_{cyclic} \frac{1}{q_1a_1 + q_2a_2 + ... + q_na_n} = \sum_{cyclic}$$

$$\left(\frac{p_1}{x_1a_1 + x_2a_2 + ... + x_ka_k} + \frac{p_2}{x_1a_2 + x_2a_3 + ... + x_ka_{k+1}} + ...+\right.$$

$$\left.+... + \frac{p_n}{x_1a_n + x_2a_1 + ... + x_ka_{k-1}}\right) \geq \left(\sum_{i=1}^{n} p_i\right)^2$$

$$\sum_{cyclic} \frac{1}{q_1a_1 + q_2a_2 + ... + q_na_n} \geq \frac{n^2 \sum_{i=1}^{n} p_i}{(x_1 + x_2 + ... + x_k) \sum_{i=1}^{n} a_i}.$$

□

References

[1] Bencze, M., *Inequalities* (manuscript), 1982
[2] Octogon Mathematical Magazine (1993-2007)
[3] Mitrinović, D.S, *Analytic Inequalities*, Springer-Verlag, Berlin, Heidelberg, New York, 1970

Harmanului 6
505600 Sacele - Négyfalu
Jud. Braşov, Romania
E-mail address: benczemihaly@yahoo.com