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Some separations results by inversion

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ABSTRACT. In this paper we solve and extend a separation problem given at the Swedish Mathematical Olympiad in 1984, using the inversion method.

1. PRELIMINARIES

First we give some construction results (e.g. [1]. [2]).

Problem 1.1. Let there be given two points A, B and a circle $\mathcal{C}$. Determine a circle $\mathcal{M}$ passing through A, B and which is tangent to $\mathcal{C}$.

Solution. Let $L \in \mathcal{C}$ be arbitrary chosen. In generally, the circle (ABL) intersects the second time $\mathcal{C}$ in K. Now, we can see that the line KL belongs to a pencil of straight lines and let F be its radical center.

In case $L = K(= T$ the requested point), KL becomes tangent TF to $\mathcal{C}$. In conclusion, T is tangent point of $\mathcal{C}$ with the pencil. If F is exterior to $\mathcal{C}$, there are two solutions. The situation $F \in \mathcal{C}$ comes when $A \in \mathcal{C}$ or $B \in \mathcal{C}$. If F lies inside $\mathcal{C}$, there are no solutions; this situation appears in case when A and B are separated by $\mathcal{C}$. $\square$

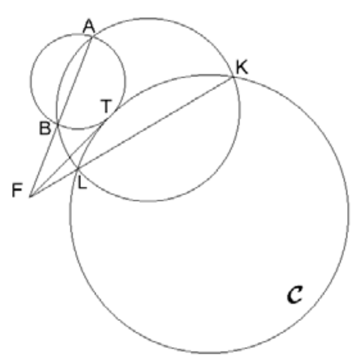

Figure 1

In order to broach this problem by inversion, let us consider an inversion **I** with pole $P \in \mathcal{C}$. In this way, the above problem converts in the following form:

Problem 1.2. Let there be given two points A', B' and a line d. Determine a circle $\mathcal{M}'$ passing through A', B' and which is tangent to d.

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Solution. This problem has a trivial elementary solution. Indeed, let us assume that $F \in A'B' \cap d$. Now we can consider the point T (two solutions) with $FT^2 = FA' \cdot FB'$, etc. $\square$

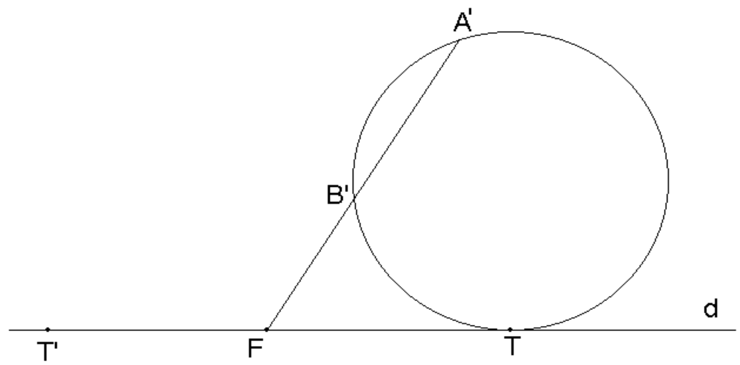

Figure 2

Other similar nice results can be found for example in [2], [3]. At the Final Round of the Swedish Mathematical Olympiad in 1984 was given the following problem:

Problem 1.3. Let A, B be two points inside a circle ω. Then there exists a circle ω_1 passing through A and B such that $\omega \cap \omega_1 = \emptyset$.

Elementary solution. Let O be the center of ω. If $OA = OB$, then take $\omega_1 = \omega'$ with center O and radius OA.

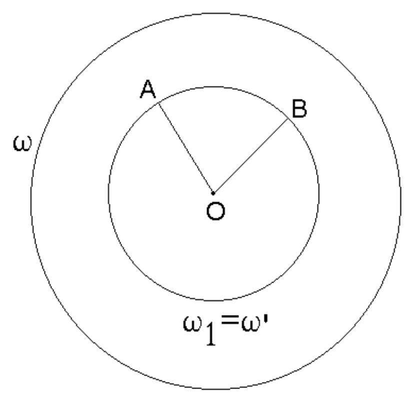

Figure 3

Further, let us assume that $OA > OB$. Let C be the second intersection point of AB with ω'. Let D be the point of the line-segment AO such that $BD \| OC$. Then take ω_1 the circle with center D and radius DA. It is internally tangent to ω', thus ω_1 lies inside ω.

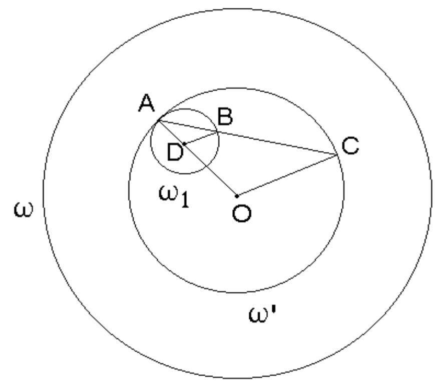

Figure 4

Solution. Let us denote by M, N the intersection points of the line AB with circle ω, such that $A \in (MB)$.

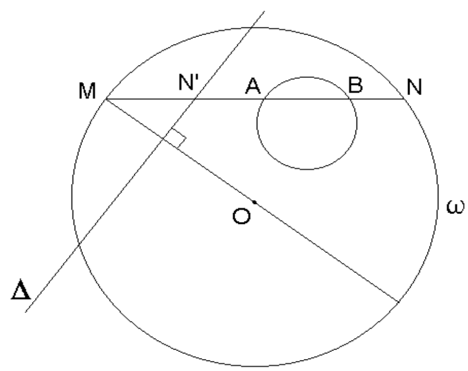

Figure 5

Let $\mathbf{I}$ be the inversion with pole M and power $k = MA \cdot MB$. We have $\mathbf{I}(A) = B$ and $\mathbf{I}(B) = A$, so all circles passing through A and B are invariant under the inversion $\mathbf{I}$. If $N' = \mathbf{I}(N)$, then $N' \in (AM)$. Let Δ be the perpendicular line on OM through N'. The inversion $\mathbf{I}$ transforms circle ω in a perpendicular line on MO. But $\mathbf{I}(N) = N'$, thus $\mathbf{I}(\omega) = \Delta$. Now, let ω' be a circle passing through A, B such that $\omega' \cap \Delta = \emptyset$. It follows that $\mathbf{I}(\omega') \cap \mathbf{I}(\Delta) = \emptyset$, which is $\omega' \cap \omega = \emptyset$. In conclusion, we can take $\omega_1 = \omega'$. □

2. The results

We also have the following separation result:

Proposition 2.1. *Let there be given A, B two points outside a circle ω. Then there exists a circle ω_2 passing through A and B such that $\omega_2 \cap \omega = \emptyset$.*

Proof. Let O be the center of the circle ω and let r be its radius. Let us consider the inversion $\mathbf{I}$ with pole O and power $k = r^2$. Obviously, ω is invariant under the inversion **I.**

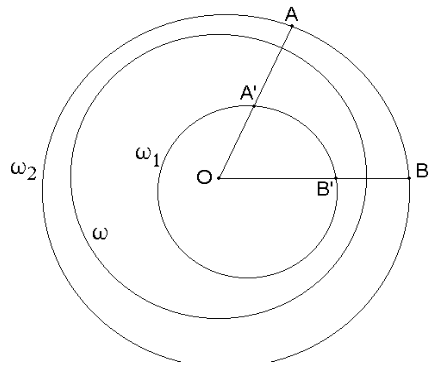

Figure 6

The points $A' = \mathbf{I}(A)$ and $B' = \mathbf{I}(B)$ lie inside the inversion circle ω. Accordingly with Problem 1.3, we can find a circle ω_1 passing through A' and B' such that $\omega_1 \cap \omega = \emptyset$. It results that $\mathbf{I}(\omega_1) \cap \mathbf{I}(\omega) = \emptyset$, which is $\mathbf{I}(\omega_1) \cap \omega = \emptyset$. In conclusion, the circle $\omega_2 = \mathbf{I}(\omega_1)$ satisfies the hypothesis. $\square$

Using these ideas from the previous solutions, we give another two results concerning intersection of two circles.

Proposition 2.2. *Let there be given two points A, B and a circle ω. Then there exists an unique circle ω_1 (or a line) orthogonal on ω such that $A, B \in \omega_1$.*

Proof. First assume that $M \in AB \cap \omega$ such that $A \in (MB)$.

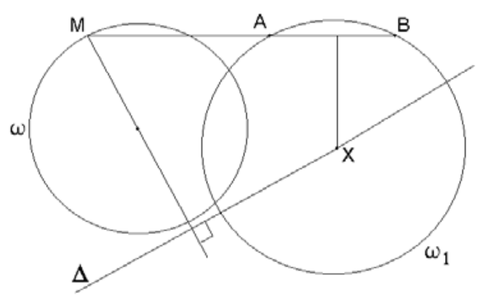

Figure 7

Let $\mathbf{I}$ be the inversion of pole M and power $k = MA \cdot MB$. Denote $\Delta = \mathbf{I}(\omega)$. The perpendicular bisector of the line-segment (AB) meets Δ in X. Then can take ω_1 the circle with center X passing through A, B. Indeed, the curves ω_1 and Δ are perpendicular. The inversion preserves the angles, thus $\omega_1 = \mathbf{I}(\omega_1)$ and $\omega = \mathbf{I}(\Delta)$ are also perpendicular. That means that the circles ω_1 and ω are orthogonal. Finally, assume that $AB \cap \omega = \emptyset$. It results that A, B lie outside ω. Let $\mathbf{I}$ be the inversion with center O and power $k = r^2$, where O is the center of ω and r is its radius. Then the points $A' = \mathbf{I}(A)$ and $B' = \mathbf{I}(B)$ lie inside ω. In particular, $A'B' \cap \omega \neq \emptyset$. We proved that we can find a circle ω_1 orthogonal on ω such that $A', B' \in \omega_1$. Then the circle $\omega_2 = \mathbf{I}(\omega_1)$ is orthogonal on ω because $\mathbf{I}(\omega) = \omega$. Moreover, $A, B \in \omega_2$ □

Proposition 2.3. *Let there be given two points A, B and a circle ω. Assume that A, B does not lie both on ω. Then there exists an unique circle ω_2 passing through A, B and meeting ω in two antipodal points.*

Proof. Let $\mathbf{I}$ be the inversion of center O and power $k = -r^2$, where O is the center of ω and r is its radius. It is well known that inversions with negative power invariate the circles which intersect the inversion circle in two antipodal points.

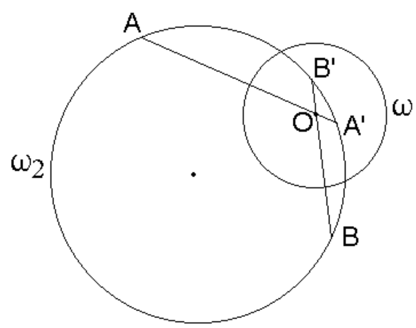

Figure 8

Let $A' = \mathbf{I}(A)$ and $B' = \mathbf{I}(B)$.
Then the circumcircle ω_2 of the quadrilateral $AB'A'A$ is invariant under the inversion $\mathbf{I}$. Hence ω_2 intersects ω in two antipodal points. □

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