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About generalization in mathematics (III). On the inclusion and exclusion principle

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ABSTRACT. Let $A_1, A_2, ..., A_n$ be finite sets and $m(A_i)$ denote the number of elements of the set A_i. In this paper we obtain a formula of type "the inclusion and exclusion principle" (Boole-Sylvester) for finding out the number of elements of the set $A_1 \Delta A_2 \Delta ... \Delta A_n$ where $A \Delta B = (A \backslash B) \cup (B \backslash A)$ is "the symmetric difference of the sets A and B":

$$m(A_1 \Delta A_2 \Delta ... \Delta A_n) =$$

$$= \sum_{i=1}^{n} m(A_i) - 2 \sum_{1 \le i < j \le n} m(A_i \cap A_j) + 2^2 \sum_{1 \le i < j < k \le n} m(A_i \cap A_j \cap A_k) - ... +$$

$$+ (-1)^{n-1} \cdot 2^{n-1} m\left(\bigcap_{i=1}^{n} A_i\right)$$

We will start from a simple problem which we will generalize in many stages, putting in evidence the importance of inductive judgment.

1. A PROBLEM

Problem 1.1. Determine the number of the natural numbers not null, smaller or equal with 1000 that are multiples of 2 or 3 or 5, but are not multiples of $2 \cdot 3$ or $2 \cdot 5$ or $3 \cdot 5$ only if they are multiples of $2 \cdot 3 \cdot 5$.

Solution. We note with A the set of the multiples of 2, with B the set of multiples of 3 and with C the set of multiples of 5 (which are not null and are smaller or equal with 1000). Then the searched number is

$$m(A) + m(B) + m(C) - 2m(A \cap B) - 2m(A \cap C) - 2m(B \cap C) + 4m(A \cap B \cap C).$$

We have:

$$m(A) = \left[\frac{1000}{2}\right] = 500, \quad m(B) = \left[\frac{1000}{3}\right] = 333, \quad m(C) = \left[\frac{1000}{5}\right] = 200,$$

$$m(A \cap B) = \left[\frac{1000}{6}\right] = 166, \quad m(A \cap C) = \left[\frac{1000}{10}\right] = 100,$$

$$m(B \cap C) = \left[\frac{1000}{15}\right] = 66, \quad m(A \cap B \cap C) = \left[\frac{1000}{30}\right] = 33.$$

From here it follows that the searched number is:

$$500 + 333 + 200 - 2 \cdot 166 - 2 \cdot 100 - 2 \cdot 66 + 4 \cdot 33 = 501$$

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Remark 1.1. This number could be obtained by writing the numbers from 1 at 1000 and eliminating the numbers that don't correspond.

2. A first generalization

An extension of the Problem 1.1. is the following

Problem 2.1. Let A, B, C be finite sets. Find out the number of elements which belong to A or to B or to C or to A and B and C, but does not belong only to A and B or to A and C or to B and C.

Solution. The searched number is given by the formula

$$m(A)+m(B)+m(C)-2m(A\cap B)-2m(A\cap C)-2m(B\cap C)+4m(A\cap B\cap C).$$

But this expression can be limited to $m(A\Delta B\Delta C)$.

Indeed, we have

$$m(A\Delta B)=m(A)+m(B)-2m(A\cap B).$$

If we use the property of distributivity of the intersection face to the symmetric difference:

$$(A\Delta B)\cap C=(A\cap C)\,\Delta\,(B\cap C)$$

we obtain

$$m(A\Delta B\Delta C)=m((A\Delta B)\,\Delta C)=m(A\Delta B)+m(C)-2m((A\Delta B)\cap C)=$$
$$=m(A)+m(B)-2m(A\cap B)+m(C)-2m((A\cap C)\,\Delta\,(B\cap C))=$$
$$=m(A)+m(B)+m(C)-2m(A\cap B)-2m(A\cap C)-2m(B\cap C)+4m(A\cap B\cap C)$$

This finding leads us to the generalization of this problem.

3. The second generalization

Theorem 3.1. *Let $A_1, A_2, ..., A_n$ be finite sets. Then we have:*

$$m(A_1\Delta A_2\Delta ...\Delta A_n)=$$
$$=\sum_{i=1}^{n}m(A_i)-2\sum_{1\le i<j\le n}m(A_i\cap A_j)+2^2\sum_{1\le i<j<k\le n}m(A_i\cap A_j\cap A_k)-...+$$
$$+(-1)^{n-1}\cdot 2^{n-1}m\left(\bigcap_{i=1}^{n}A_i\right) \tag{3.1}$$

Proof. We will prove through induction.

For $n=2$ we have:

$$m(A_1\Delta A_2)=m(A_1)+m(A_2)-2m(A_1\cap A_2)$$

Suppose the sentence is true for p and we will prove that it is true for $p+1$, too. We have

$$m(A_1\Delta A_2\Delta ...\Delta A_{p+1})=m((A_1\Delta A_2\Delta ...\Delta A_p)\,\Delta A_{p+1})=m(A_1\Delta A_2\Delta ...\Delta A_p)+$$
$$+m(A_{p+1})-2m((A_1\Delta A_2\Delta ...\Delta A_p)\cap A_{p+1}).$$

Using the hypothesis of induction and distributive law of intersection face to the symmetric difference, we obtain

$$m\left(A_1\Delta A_2\Delta...\Delta A_{p+1}\right)=$$

$$=\sum_{i=1}^{p}m\left(A_i\right)-2\sum_{1\leq i<j\leq p}m\left(A_i\cap A_j\right)+2^2\sum_{1\leq i<j<k\leq p}m\left(A_i\cap A_j\cap A_k\right)-...+$$

$$+\left(-1\right)^{p-1}2^{p-1}m\left(\bigcap_{i=1}^{p}A_i\right)+m\left(A_{p+1}\right)-$$

$$-2m\left(\left(A_1\cap A_{p+1}\right)\Delta\left(A_2\cap A_{p+1}\right)\Delta...\Delta\left(A_p\cap A_{p+1}\right)\right)]$$

According to the hypothesis of induction we have:

$$m((A_1\cap A_{p+1})\Delta\left(A_2\cap A_{p+1}\right)\Delta...\Delta\left(A_p\cap A_{p+1}\right))=$$

$$=\sum_{i=1}^{p}m\left(A_i\cap A_{p+1}\right)-2\sum_{1\leq i<j\leq p}m\left(A_i\cap A_j\cap A_{p+1}\right)$$

$$+...+\left(-1\right)^{p-1}2^{p-1}m\left(\bigcap_{i=1}^{p}\left(A_i\cap A_{p+1}\right)\right).$$

Using the idempotence of intersection we have:

$$(A_i\cap A_{p+1})\cap(A_j\cap A_{p+1})=(A_i\cap A_j\cap A_{p+1}),...,\bigcap_{i=1}^{p}(A_i\cap A_{p+1})=\bigcap_{i=1}^{p+1}A_i.$$

Regrouping the terms we obtain

$$m\left(A_1\Delta A_2\Delta...\Delta A_p\Delta A_{p+1}\right)=$$

$$=\sum_{i=1}^{p+1}m\left(A_i\right)-2\sum_{1\leq i<j\leq p+1}m\left(A_i\cap A_j\right)+2^2\sum_{1\leq i<j<k\leq n}m\left(A_i\cap A_j\cap A_k\right)-...+$$

$$...+\left(-1\right)^{p}\cdot 2^{p}m\left(\bigcap_{i=1}^{p+1}A_i\right)$$

□

So ”the inclusion and exclusion principle” for the symmetric difference, formula (3.1) is proved.

From this theorem we deduce that if an element belongs to the symmetric difference of n sets, then the maximum number of sets to which it belongs is and odd number.

We will prove this property directly in the next theorem.

Theorem 3.2. *Let $A_1, A_2, ..., A_n$ be sets. If $x\in A_1\Delta A_2\Delta...\Delta A_n$ then the biggest number of sets to which x belongs is an odd number.*

Proof. We will prove through induction after n.

For $n = 2$ we have that if : $x \in A_1 \Delta A_2$ then $x \in A_1$ or $x \in A_2$, but it does not belong to $A_1 \cap A_2$ because

$$A_1 \Delta A_2 = (A_1 \cup A_2) \setminus (A_1 \cap A_2)$$

Suppose the sentence is true for k and we will prove that it is true for $k+1$, too. If

$$x \in A_1 \Delta A_2 \Delta ... \Delta A_k \Delta A_{k+1}$$

then

$$x \in (A_1 \Delta A_2 \Delta ... \Delta A_k) \Delta A_{k+1}.$$

Hence we have

$$x \in (A_1 \Delta A_2 \Delta ... \Delta A_k) \cup A_{k+1}$$

and

$$x \notin (A_1 \Delta A_2 \Delta ... \Delta A_k) \cap A_k.$$

We obtain that $x \in A_1 \Delta A_2 \Delta ... \Delta A_k$ or $x \in A_{k+1}$. If $x \in A_1 \Delta A_2 \Delta ... \Delta A_k$ then from the induction hypothesis the maximum number of sets to which x belongs is odd. If $x \in A_{k+1}$ the theorem is proved □

4. Particular cases

Problem 4.1. Let $p_1, p_2, ..., p_k$ be prime natural numbers, $n \in \mathbb{N}$, $p_i < n, i = \overline{1,k}$; $k > 3$. Find the number of all non zero natural numbers, smaller or equal with n, which have divisors of the form

$$p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_q},\ q \leq k$$

in which the maximum number of prime numbers can be only an odd number.

Solution. We note with A_i the set of p_i multiples smaller or equal with n.Then the requested numbers can belong to maximum to an odd number of A_i sets.

From Theorem 3.2 it follows that the searched number is given by $m(A_1 \Delta A_2 \Delta ... \Delta A_n)$.

For $n = 1000$ and $p_1 = 2,\ p_2 = 3,\ p_3 = 5$ we obtain Problem 1.1.

5. The third generalization

Let A be a bounded subset of $\mathbb{R}^2$, measurable and we note with $m(A)$ its measure. For example A can be a rectangular surface and $m(A)$ its area. The following properties are known.

Lemma 5.1. ([1]) *If A and B are bounded and measurable sets in $\mathbb{R}^2$, then the sets $A \backslash B, A \cup B,\ A \cap B$ are also measurable in $\mathbb{R}^2$.*

From here it follows if A and B are bounded and measurable sets in $\mathbb{R}^2$, then $A \Delta B$ is a bounded and measurable set, because

$$A \Delta B = (A \backslash B) \cup (B \backslash A)$$

Using these properties we can extended Theorem 3.1. at n measurable sets in plane.

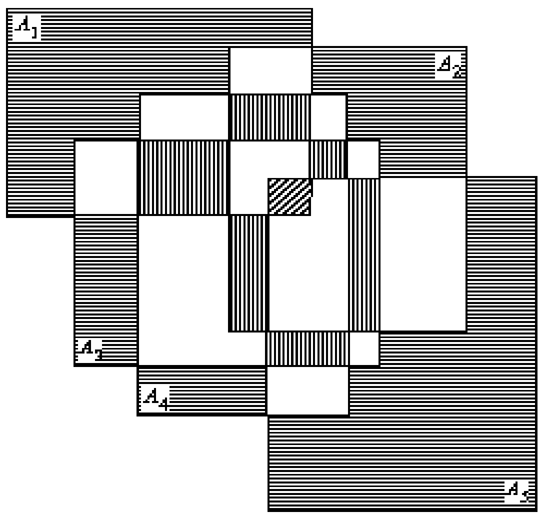
Theorem 5.1. *If* $A_1, A_2, ..., A_n$ *are bounded and measurable in* $\mathbb{R}^2$ *then*

$$m\left(A_1 \Delta A_2 \Delta ... \Delta A_n\right) =$$

$$= \sum_{i=1}^{n} m\left(A_i\right) - 2 \sum_{1\leq i<j\leq n} m\left(A_i \cap A_j\right) + 2^2 \sum_{1\leq i<j<k\leq n} m\left(A_i \cap A_j \cap A_k\right) - ...$$

$$+ \left(-1\right)^{n-1} \cdot 2^{n-1} m\left(\bigcap_{i=1}^{n} A_i\right)$$

The proof is the same as for Theorem 3.1.
In the next figure we illustrate these properties for five sets.



Remark 5.1. Obviously, we could continue the extension in other spaces with different types of measures, but we stop here at elementary mathematics.

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