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A new weighted Erdős-Mordell type inequality

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ABSTRACT. A new weighted Erdős-Mordell type inequality involving interior point of a triangle is established. By it's application, some interesting geometric inequalities are derived.

1. INTRODUCTION

Throughout the paper we assume $\triangle ABC$ be a triangle, and denote by a, b, c its sides' lengths, by Δ its area. Let P be an interior point. Denote Δ_1, Δ_2, Δ_3 the area of $\triangle BPC$, $\triangle CPA$, $\triangle APB$. Denote R_a, R_b, R_c the circumradii of the triangles BPC, CPA, APB. Let R_1, R_2, R_3 be the distances from P to A, B, C, and let r_1, r_2, r_3 be the distances of P from the sides AB, BC, CA. Denote by w_1, w_2, w_3 the bisectors of the angles BPC, CPA, APB. Then the following theorem holds.

Theorem 1.1.

$$R_1 + R_2 + R_3 \geq 2\left(r_1 + r_2 + r_3\right) \tag{1.1}$$

The inequality (1.1) is sharp: equality holds if and only if the triangle is equilateral and the point P is its center. This is the famous Erdős-Mordell inequality. It was conjectured by Erdős in 1935 [1], and was first proved by Mordell in 1937 [2].

In the paper [3], D.S. Mitrnović at al. noted some generalizations of Erdős-Mordell inequality in 1989. Among their results is the following three-variable quadratic Erdős-Mordell type inequality:

Theorem 1.2. *If x, y, z are three real numbers, then for any point P inside the triangle ABC, we have*

$$x^2R_1 + y^2R_2 + z^2R_3 \geq 2\left(yzr_1 + zxr_2 + xyr_3\right) \tag{1.2}$$

with equality holding if and only if $x = y = z$ and P is the center of equilateral triangle ABC.

In this note we give a new weighted Erdős-Mordell type inequality.

2. MAIN RESULT

In order to prove Theorem 2.1 below, we need the following lemma.

Lemma 2.1. *For any triangle ABC and $x, y, z \in R$ we have*

$$x^2sin^2A + y^2sin^2B + z^2sin^2C \leq \frac{1}{4}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{zy}{z}\right)^2. \tag{2.1}$$

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Proof. The inequaliyty (2.1) can be obtained from O. Kooi's inequality [4]:

$$(x+y+z)^2R^2 \geq yza^2 + zxb^2 + xyc^2$$

where R is the circumradius of the triangle ABC. □

Theorem 2.1. *For an arbitrary point P inside $\triangle ABC$, let R_a, R_b, R_c be the circumradii of triangles BPC, CPA, APB, and w_1, w_2, w_3 be the bisectors of the angles BPC, CPA, APB. If $x > 0$, $y > 0$, $z > 0$, then*

$$\frac{xw_1}{\sqrt{R_bR_c}} + \frac{yw_2}{\sqrt{R_cR_a}} + \frac{zw_3}{\sqrt{R_aR_b}} \leq \frac{1}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right) \tag{2.2}$$

with equality holding if and only if $x = y = z$ and P is the center of equilateral triangle ABC.

Proof. Let $\angle BPC = \alpha$, $\angle CPA$=β, $\angle APB$=γ. Obviously $0 < \alpha$, β, $\gamma < \pi$ and α+β +γ =2π. By using law of sines and bisectors formula we have

$$b = 2R_b sin\beta$$
$$c = 2R_c sin\gamma$$
$$w_1 = \frac{2R_2R_3}{R_2 + R_3} cos\frac{\alpha}{2} \leq \sqrt{R_2R_3} cos\frac{\alpha}{2}$$
$$\frac{w_1}{\sqrt{R_bR_c}} \leq \sqrt{\frac{R_2R_3}{R_bR_c}} \cos\frac{\alpha}{2}$$
$$= 2\sqrt{\frac{R_2R_3 \sin\beta \sin\gamma}{bc}} \cos\frac{\alpha}{2}$$
$$= 2\sqrt{\frac{\Delta_1 \sin\beta \sin\gamma \sin A}{\Delta \sin\alpha}} \cos\frac{\alpha}{2}$$

Because

$$\sqrt{\sin\beta\sin\gamma} \leq \frac{1}{2}(\sin\beta + \sin\gamma)$$
$$= \sin\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2} \leq \sin\frac{\alpha}{2}$$
$$\sqrt{\frac{\sin\beta\sin\gamma}{\sin\alpha}}\cos\frac{\alpha}{2} \leq \frac{1}{2}\sqrt{\sin\alpha}$$

we have

$$\frac{w_1}{\sqrt{R_bR_c}} \leq \sqrt{\frac{\Delta_1}{\Delta}\sin A\sin\alpha} \tag{2.3}$$

Let $A' = \pi - \alpha, B' = \pi - \beta, C' = \pi - \gamma$, then

$$\sqrt{\sin A\sin\alpha} = \sqrt{\sin A\sin A'} \leq \frac{1}{2}(\sin A + \sin A') = \sin\frac{A+A'}{2}\cos\frac{A-A'}{2} \leq \sin\frac{A+A'}{2} \tag{2.4}$$

From (2.3) and (2.4) we have

$$\frac{xw_1}{\sqrt{R_bR_c}} \leq \sqrt{\frac{\Delta_1}{\Delta}} x\sin\frac{A+A'}{2} \tag{2.5}$$

By using the same method we have

$$\frac{yw_2}{\sqrt{R_cR_a}} \le \sqrt{\frac{\Delta_2}{\Delta}}y\sin\frac{B+B'}{2} \tag{2.6}$$

$$\frac{xw_3}{\sqrt{R_aR_b}} \le \sqrt{\frac{\Delta_3}{\Delta}}z\sin\frac{C+C'}{2} \tag{2.7}$$

Combining (2.5), (2.6), (2.7) and by Cauchy's inequality we have

$$\begin{aligned}
&\frac{xw_1}{\sqrt{R_bR_c}} + \frac{yw_2}{\sqrt{R_cR_a}} + \frac{xw_3}{\sqrt{R_aR_b}} \\
&\le \sqrt{\frac{\Delta_1}{\Delta}}x\sin\frac{A+A'}{2} + \sqrt{\frac{\Delta_2}{\Delta}}y\sin\frac{B+B'}{2} + \sqrt{\frac{\Delta_3}{\Delta}}z\sin\frac{C+C'}{2} \\
&\le \sqrt{(\frac{\Delta_1}{\Delta}+\frac{\Delta_2}{\Delta}+\frac{\Delta_3}{\Delta})(x^2\sin^2\frac{A+A'}{2}+y^2\sin^2\frac{B+B'}{2}+z^2\sin^2\frac{C+C'}{2})} \\
&= \sqrt{x^2\sin^2\frac{A+A'}{2}+y^2\sin^2\frac{B+B'}{2}+z^2\sin^2\frac{C+C'}{2})}
\end{aligned}$$

Let

$$\theta = \frac{A+A'}{2}, \phi = \frac{B+B'}{2}, \varphi = \frac{C+C'}{2}.$$

Obviously $0 < \theta, \phi, \varphi < \pi$ and $\theta + \phi + \varphi = \pi$, so θ, ϕ, φ can be angles of a triangle $A_1B_1C_1$. Applying Lemma 2.1 for the triangle $A_1B_1C_1$ we obtain

$$x^2\sin^2\theta + y^2\sin^2\phi + z^2\sin^2\varphi \le \frac{1}{4}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{zy}{z}\right)^2$$

with equality holding if and only if $x = y = z$ and P is the center of equilateral triangle ABC. The proof of Theorem 2.1 is completed. □

3. SOME APPLICATIONS

In this section we give some applications of Theorem 2.1.

Noticed $r_1 \le w_1$ etc., we have

$$\frac{xr_1}{\sqrt{R_bR_c}} + \frac{yr_2}{\sqrt{R_cR_a}} + \frac{zr_3}{\sqrt{R_aR_b}} \le \frac{1}{2}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right) \tag{3.1}$$

By using AM-GM inequality we have $\sqrt{R_bR_b} \le \frac{1}{2}(R_b+R_b)$, then from (3.1) we have

$$\frac{xr_1}{R_b+R_c} + \frac{yr_2}{R_c+R_a} + \frac{zr_3}{R_a+R_b} \le \frac{1}{4}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right) \tag{3.2}$$

By the same way of (3.2), the following inequality holds

$$\frac{xw_1}{R_b+R_c} + \frac{yw_2}{R_c+R_a} + \frac{zw_3}{R_a+R_b} \le \frac{1}{4}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right) \tag{3.3}$$

Let $x = y = z = 1$ in (3.2), then

$$\frac{r_1}{R_b+R_c} + \frac{r_2}{R_c+R_a} + \frac{r_3}{R_a+R_b} \le \frac{3}{4} \tag{3.4}$$

In fact (3.4) was conjectured by Jian Liu in [5] and here we obtained a proof.

Theorem 3.1. *If* $x > 0, y > 0, z > 0$, *then*

$$x^2R_a + y^2R_b + z^2R_c \geq 2\left(yzw_1 + zxw_2 + xyw_3\right) \tag{3.5}$$

Proof. Alter $x \to x'\sqrt{R_bR_c}, y \to y'\sqrt{R_cR_a}, z \to z'\sqrt{R_aR_b}\,(x, y, z > 0)$ in (2.2) we obtain

$$x'w_1 + y'w_2 + z'w_3 \leq \frac{1}{2}\left(\frac{y'z'}{x'}R_a + \frac{z'x'}{y'}R_b + \frac{x'y'}{z'}R_c\right) \tag{3.6}$$

and then let $\frac{y'z'}{x'} = x^2, \frac{z'x'}{y'} = y^2, \frac{x'y'}{z'} = z^2$ in (3.6), then (3.5) is obtained □

Inequality (3.5) is similar to (1.2) that was conjectured by Jian Liu in [6]. Obviously,

$$x^2R_a + y^2R_b + z^2R_c \geq 2\left(yzr_1 + zxr_2 + xyr_3\right) \tag{3.7}$$

Let $x = y = z = 1$ in (3.5) and (3.7), then we have

$$R_a + R_b + R_c \geq 2\left(w_1 + w_2 + w_3\right) \tag{3.8}$$

and also

$$R_a + R_b + R_c \geq 2\left(r_1 + r_2 + r_3\right) \tag{3.9}$$

Inequality (3.9) is similar to (1.1).
Let $x = y = z = 1$ in (2.2) and by AM-GM inequality we have

$$R_aR_bR_c \geq 8w_1w_2w_3 \tag{3.10}$$

and

$$R_aR_bR_c \geq 8r_1r_2r_3. \tag{3.11}$$

Let $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$ in (3.7), then we have

$$\frac{R_a}{a^2} + \frac{R_b}{b^2} + \frac{R_c}{c^2} \geq \frac{1}{R}. \tag{3.12}$$

REFERENCES

[1] Erdős, P., Problem 3740. *Amer. Math. Monthly*, **42** (1935), 396

[2] Mordell L.J., Barrow, D.F., Solution 3740, *Amer. Math. Monthly.* **44**(1937), 252-254

[3] D.S. Mitrinovič, J.E. Pečarć and V. Voloneć, Recent Advances in Geometric Inequalities, Dordrecht Netherlands Kluwer, 1989

[4] Bottema O. et al. *Geometric Inequalities*, Groningen Wolters-Noordhoff, 1969

[5] Jian Liu, *A hundred unsolved Triangle inequality problems*, Geometric inequalities in China, Nanjing Jiangsu Education Press, 1996

[6] Jian Liu, *Two Applications of A Geometric Inequality*, Journal of Kaifeng University, **1**(2004), 88-91

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