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Sublinear mappings and metric regularity

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ABSTRACT.
The metric regularity is a central concept in variational analysis. This concept is frequently used for the study of solutions to some generalized equations, to variational inequalities and to parametrized constraints systems. Fundamental theorems of this field are Eckart-Young's, Robinson-Ursescu's and Lyusternyk-Grave's theorem. They have applications in single valuedness functions theory and also in set-valued mappings theory. The purpose of this article is to analize the metric regularity property of some sublinear applications.

1. INTRODUCTION

Let X and Y be real Banach spaces with norms both denoted by $\|\cdot\|$ and closed unit balls $\mathbb{B}_X$ and $\mathbb{B}_Y$. Let F be a mapping from X to Y, by which we will generally mean a set-valued mapping, indicated by $F : X \rightrightarrows Y$, having inverse $F^{-1} : Y \rightrightarrows X$ with $x \in F^{-1}(y)$ if and only if $y \in F(x)$, and having effective graph, domain and range sets given respectively by:

$$\operatorname{gph} F = \{(x, y) | y \in F(x)\}, \;\; \operatorname{dom} F = \{x | F(x), \text{ is nonempty}\}, \;\; \operatorname{rge} F = \operatorname{dom} F^{-1}.$$

If F is single-valued we denote it $F : X \to Y$. The terminology of "generalized equations" has the form $y \in F(x)$, where x is a solution for a given y. We also have $F^{-1}(y)$ nonempty if and only if $y \in \operatorname{rge} F$. It is natural that the solutions are changed by the restrictions on y.

It is known that the equation $Ax = y$, where $A : \mathbb{R}^n \to \mathbb{R}^n$, has unique solution if A is nonsingular. In this case it is known in what measure does perturbation of A lead to the same property. This result is contained in Eckart-Young Theoreme (see [4]):

$$\inf \left\{ \|B\| \;\middle|\; A + B \text{ singular} \right\} = \frac{1}{\|A^{-1}\|}. \tag{1.1}$$

A similar result is obtained also in the metric regularity domain, or in the case of a continuous linear mappings $F : X \to Y$ for $\dim Y = \dim X < \infty$. This kind of characterization will be stated in the following paragraphs. If $G : X \to Y$ and $\overline{y} = G(\overline{x})$ then, the Lipschitz module is defined by

$$\operatorname{lip} G(\overline{x}) := \limsup_{\substack{x, x' \to \overline{x} \\ x \neq x'}} \frac{\|G(x) - G(x')\|}{\|x - x'\|}. \tag{1.2}$$

The condition for $\operatorname{lip} G(\overline{x}) < \infty$ is equivalent to the fact that G is Lipschitz continuous around $\overline{x}$. When $G(x) = Ax + a$, which means a continuous linear mapping, it is obvious that $\operatorname{lip} G(\overline{x}) = \|A\|$.

2. BACKGROUND IN METRIC REGULARITY

The concept of metric regularity appeared in an indirect way in 1930 at Lyuster-nik and in a context more explicit in the works of Graves in 1950, evolving in optimization problems in an accelerated way after 1960 (see [5]). By $\mathbb{B}_r(a)$ we will understand the closed ball of center a and the radius r, $B_r(a) = a + r\mathbb{B}_X$ and the distance from the x point to a set C is denoted

$$d(x, C) = \inf \left\{ \|x - x'\| \;\middle|\; x' \in C \right\}.$$

Definition 2.1. ([4]) A mapping $F : X \rightrightarrows Y$ is metrically regular at $\overline{x}$ for $\overline{y}$ if $\overline{y} \in F(\overline{x})$ and there exists $k \in [0, \infty)$ along with neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$d(x, F^{-1}(y)) \leq k d(y, F(x)), \tag{2.3}$$

for all $x \in U$, $y \in V$.

The infimum of these k's for which (2.3) holds is the modulus of metric regularity, denoted by $\operatorname{reg} F(\overline{x} \,|\, \overline{y})$. The absence of metric regularity is denoted as $\operatorname{reg} F(\overline{x} \,|\, \overline{y}) = \infty$. The inequality (2.3) has directly use providing an estimate for how for a point x is from being a solution to the general equation for F and data y. The expression $d(y, F(x))$ measures the "residual" when $y \notin F(x)$. In the case of $F(x) = Ax + a$, where A is an $n \times n$ matrix and $a \in \mathbb{R}^n$, the modulus of metric regularity is the same for any $\overline{x} \in X$, $\operatorname{reg} F(\overline{x} \,|\, \overline{y}) = \|A^{-1}\|$, for A nonsingular.

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Definition 2.2. ([3]) For a multifunction $F : X \rightrightarrows Y$ we say that, $F^{-1} : Y \rightrightarrows X$ has the Aubin property at $\overline{y}$ for $\overline{x}$, when there exists $k \in [0, \infty)$ along with neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$F^{-1}(y') \cap U \subset F^{-1}(y) + k\|y' - y\|\mathbb{B}_X, \quad \text{for all } \; y, y' \in V. \tag{2.4}$$

An equivalent characterization with metric regularity is given by Theorem 2.1 ([4]).

Theorem 2.1. *For a multifunction $F : X \rightrightarrows Y$, let $\overline{y} \in F(\overline{x})$. Then F is metrically regular at $\overline{x}$ for $\overline{y}$ if and only if it's inverse $F^{-1} : Y \rightrightarrows X$ has the Aubin property at $\overline{y}$ for $\overline{x}$. The infimum of this k's is denoted $\operatorname{lip} F^{-1}(\overline{y}\,|\,\overline{x})$. The two Definition 2.1 and 2.2 are equivalent and*

$$\operatorname{lip} F^{-1}(\overline{y}\,|\,\overline{x}) = \operatorname{reg} F(\overline{x}\,|\,\overline{y}). \tag{2.5}$$

Another way of looking at the regularity modulus, $\operatorname{reg} F(\overline{x}\,|\,\overline{y})$ is by the property of F being linearly open, or locally surjective at $\overline{x}$ for $\overline{y}$ which refers to the existence of $k \in (0, \infty)$ and a neighborhood O of $\overline{y}$ so that

$$F(x + \operatorname{int} kr\mathbb{B}_X) \supset [F(x) + \operatorname{int} r\mathbb{B}_Y] \cap O, \tag{2.6}$$

for all x close to $\overline{x}$ and $r > 0$. These two are equivalent to the metric regularity of F at $\overline{x}$ for $\overline{y}$ with the same range of values k, and thus it yields for the regularity modulus a third formula

$$\operatorname{reg} F(\overline{x}\,|\,\overline{y}) = \inf\{k \in (0, \infty) \,|\, (2.4) \text{ holds}\}.$$

Again, details can be found in [3]. The general and local regularity criteria are proved in [5].

Corollary 2.1. ([3]) *If $F \in L(X, Y)$ (space of continuous linear mappings), then for $(\overline{x}, \overline{y}) \in \operatorname{gph} F$ we have:*

$$\begin{aligned} \operatorname{reg} F(\overline{x}\,|\,\overline{y}) &= \inf\left\{k \in (0, \infty) \middle|\, kF(\mathbb{B}_X) \supset \operatorname{int} \mathbb{B}_Y\right\} = \\ &= \sup\left\{d(0, F^{-1}(y)) \,\middle|\, y \in \mathbb{B}_Y\right\}. \end{aligned} \tag{2.7}$$

Proof. Since F is linear, by (2.6) we have:

$$F(x) + rk \operatorname{int} F(\mathbb{B}_X) \supset [F(x) + \operatorname{int} r\mathbb{B}_Y] \cap O$$

and so it holds $kF(\mathbb{B}_X) \supset \operatorname{int} \mathbb{B}_Y$. On the other hand the relation (2.3) can by written as $d(0, F^{-1}(y) - x) \leq kd(y - F(x), 0)$. Since F is linear $F^{-1}(y) - x = F^{-1}(y - y')$, where $y' = F(x)$ and so $d(0, F^{-1}(y - y')) \leq k\|y - y'\|$, or $d(0, F^{-1}(z)) \leq k\|z\|$, where z is from the neighborhood of O. It also results that the inequality holds for all $z \in Y$. □

Corollary 2.2. ([3]) *If $\dim X = \dim Y < \infty$ and $F \in L(X, Y)$, then $\operatorname{reg} F = \|F^{-1}\|$. Moreover $\operatorname{reg} F < \infty$ if and only if F is a surjection.*

Proof. From relation (2.7) it is obvious that

$$\sup\left\{\|F^{-1}(y)\| \,\middle|\, \|y\| \leq 1\right\} = \|F^{-1}\| = \operatorname{reg} F,$$

so it is the same for all $\overline{x} \in X$. By the Banach's open mapping principle we have that F is onto. □

Corollary 2.3. *Consider $X = Y = \mathbb{R}^n$, and $A \in L(X, Y)$, $(\overline{x}, \overline{y}) \in \operatorname{gph} F$. If A has the property of metric regularity at $\overline{x}$ for $\overline{y}$, the A is noninjective. If A is a square matrix of n order, the equation $Ax = y$ has at least two solutions.*

Proof. By continuity, for $\overline{y} \in A(\overline{x})$, there exist neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that for all $x \in U$ we have $A(x) \in V$. Consider the neighborhoods in the definition of metric regularity. Thus (2.3) is satisfied for any $k > 0$, since $d(y, A(x)) = 0$. This leads to $d(x, A^{-1}(y)) = 0$ for $x \in A^{-1}(y), y \in A(x)$. If we choose $y = \overline{y}$ we also have the $x \neq \overline{x}$ is a solution, which shows that A is not injective. By $\operatorname{reg} A(\overline{x}\,|\,\overline{y}) = \|A^{-1}\| < \infty$, we deduce that A is onto (Corollary 2.2). The system $Ax = y$, has at least two solutions. Obviously, if A is inversable, then the solution is unique. □

Theorem 2.2. ([6]). *If $F : \mathbb{R} \to \mathbb{R}$ and $\overline{x} \in \mathbb{R}$ with $F'(\overline{x}) \neq 0$, then $\operatorname{reg} F(\overline{x}\,|\,\overline{y}) = \dfrac{1}{|F'(\overline{x})|}$, where $\overline{y} = F(\overline{x})$, and if $F'(\overline{x}) = 0$, the metric regularity property does not hold.*

Corollary 2.4. *If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ so that $\operatorname{reg} f(\overline{x}\,|\,\overline{y}) = \operatorname{reg} g(\overline{x}\,|\,\overline{y}_1)$, for every $\overline{x} \in I$ (interval) with $\overline{y}_1 = g(\overline{x})$ and if f, g are differentiable of I, with non zero derivatives for all $x \in I$, then on the interval I the two functions satisfy $f(x) + \varepsilon g(x) =$ constant, where $\varepsilon \in \{-1, 1\}$.*

Proof. By $\operatorname{reg} f(\overline{x}\,|\,\overline{y}) = \operatorname{reg} g(\overline{x}\,|\,\overline{y})$, for all $\overline{x} \in I$ it results that $|f'(\overline{x})| = |g'(\overline{x})|$ and from $f'(\overline{x}) \neq 0$, $g'(\overline{x}) \neq 0$ we deduce that on this interval the sign is constant. If $f'(x) > 0$, $g'(x) > 0$ for all $x \in I$ it results that $f'(x) = g'(x)$, and thus $f(x) - g(x) =$ constant. If $f'(x) < 0$, $g'(x) < 0$ we have the same conclusion, and for the case $f'(x) > 0$, $g'(x) < 0$, we have $f(x) + g(x) =$ constant. □

Remark 2.1. The same conclusion follows by the Lagrange mean value theorem.
Generally, from $\operatorname{reg} F(\overline{x}\,|\,\overline{y}) = \operatorname{reg} G(\overline{x}\,|\,\overline{y})$ we can not get the conclusion that the two functions coincide.

Example 2.1. If $F, G : \mathbb{R} \to \mathbb{R}$, $F(x) = ax + b$, $G(x) = ax + c$, where $b \neq c$, then $\operatorname{reg} F(\overline{x} \,|\, \overline{y}) = \dfrac{1}{|a|} = \operatorname{reg} G(\overline{x} \,|\, \overline{y})$.

Theorem 2.3. (Robinson-Ursescu [3]). *For a mapping $F : X \rightrightarrows Y$ and $(\overline{x} \,|\, \overline{y}) \in \operatorname{gph} F$, if F has a closed convex graph, then F is metrically regulated at $\overline{x}$ for $\overline{y}$ if and only if $\overline{y} \in \operatorname{int} \operatorname{rge} F$.*

Remark 2.2. Consider $X = \mathbb{R}^n$, $Y = \mathbb{R}$ and $F(x) = f(x) + \mathbb{R}_+$, where $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $\mathbb{R}_+ = [0, \infty)$. Suppose f is a l.s.c. proper convex function. We have $F^{-1}(\alpha) = \left\{x \in \mathbb{R}^n \,\middle|\, f(x) \leq \alpha\right\} \equiv \operatorname{lev}_{\leq \alpha} f$ and $\operatorname{gph} F = \operatorname{epi} f$, so convex and closed. According to the Theorem 2.3, F is metrically regular at $\overline{x} \in \operatorname{dom} f$ for $\overline{y} = f(\overline{x})$, if and only if $f(\overline{x}) \in \operatorname{int} \operatorname{rge} F = (\inf f, +\infty)$. In other words if and only if there exists a point $x^0 \in \mathbb{R}^n$, such that $f(x^0) < f(\overline{x})$ (Slater constraint qualification [1]).

For $n = 1$, we have that at points of extreme of a function f where f is differentiable too, function f does not have the metric regularity property. In the points of minimum, it does not have the metric regularity property for convex functions, and similarly, in the points of maximum for concave functions.

If $f : \mathbb{R} \to \mathbb{R}$, $f'(\overline{x}) = 0$ and $\overline{y} = f(\overline{x})$, then on a neighborhood V of $\overline{x}$ and U of $\overline{y}$ we have:

$$d(x, f^{-1}(y)) = \frac{1}{|f'(c)|}\, d(y, f(x)),\ x \in V,\ y \in U,$$

c lies between x and $f^{-1}(y)$.

The metric regularity module is $\operatorname{reg} f(\overline{x}) = \dfrac{1}{|f'(\overline{x})|}$, so the function does not satisfy the metric regularity property on $\overline{x}$.

3. Applications of the metric regularity principle

Results characterizing the norm on a Hilbert spaces, Minkovski's function p_Y, and the support function of a set σ_C, are given in this paragraph.

Let H be a Hilbert space, with the scalar product $\langle \cdot, \cdot \rangle$ and the norm generated by $h : H \to R$, $h(x) = \langle x, x \rangle = \|x\|^2$. For $a \in H \backslash \{0\}$ the Fréchét differential of this function on a, is given by

$$Dh(a)x = \begin{cases} 2\,\langle a, x \rangle, & \text{when } \ K = \mathbb{R} \\ 2 \operatorname{Re} \langle a, x \rangle, & \text{when } \ K = \mathbb{C}. \end{cases} \tag{3.8}$$

For a Fréchét differential mapping, we have the following result:

Theorem 3.4. (Lyusternik-Graves [3]). *For any continuous Fréchét differentiable mapping $F : X \to Y$ and for any $(\overline{x}, \overline{y}) \in \operatorname{gph} F$, one has*

$$\operatorname{reg} F(\overline{x} \,|\, \overline{y}) = \operatorname{reg} DF(\overline{x}). \tag{3.9}$$

Thus F is metrically regular at $\overline{x}$ for $\overline{y} = F(\overline{x})$ if and only if $DF(\overline{x})$ is surjective.

The same result is expressed in [2] and [7].

Application 3.1. If $h : \mathbb{R}^n \to \mathbb{R}$, $h(x) = \langle x, x \rangle = \|x\|^2$, then

$$\operatorname{reg} h(\overline{a} \,|\, \overline{y}) = \operatorname{reg} Dh(\overline{a})(\overline{y}) = h(\overline{a}) = \langle \overline{a} \,|\, \overline{a} \rangle = \|\overline{a}\|^2 = \sum_1^n a_i^2.$$

Since $Dh(\overline{a}) \in L(\mathbb{R}^n, \mathbb{R})$, then from (2.7) we have:

$$\operatorname{reg} Dh(\overline{a}) = \|Dh^{-1}(\overline{a})\| = \sup\left\{\|Dh^{-1}(\overline{a})(y)\| \,\middle|\, |y| \leq 1\right\}.$$

If $a = Dh^{-1}(\overline{a})(y)$ it results

$$Dh(\overline{a})(a) = y \Rightarrow \sum_1^n \overline{a}_i \cdot a_i = y$$

where $\overline{a} = (\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n)$, $a = (a_1, a_2, \ldots, a_n)$ and $\overline{a} \neq 0 \in \mathbb{R}^n$. For $y = 0$ we have $\displaystyle\sum_1^n \overline{a}_i \cdot a_i = 0$. Since $\overline{a} \neq 0$, exists $j, i \in \{1, 2, \ldots, n\}$ such that $\overline{a}_j, \overline{a}_i \neq 0$. Consider $a = \left(0, 0, \ldots, \dfrac{m}{a_i}, \ldots, \dfrac{-m}{a_j}, \ldots, 0\right)$, and $\langle \overline{a}, a \rangle = 0$, $\|a\| = m\left(\dfrac{1}{a_i^2} + \dfrac{1}{a_j^2}\right)^{1/2} \Rightarrow \|D^{-1}h(\overline{a})\| = \infty = \operatorname{reg} h(\overline{a} \,|\, \overline{y})$, so function h does not have the metric regularity property. If $\overline{a}_i \neq 0$ and $\overline{a}_k = 0$, for $k \neq i$, consider $a = (0, 0, \ldots, m, 0, \ldots, 0)$, $\overline{a}_k = m$, $m > 0$, and $\|a\| = m \Rightarrow \|Dh^{-1}(\overline{a})\| = \infty = \operatorname{reg} h(\overline{a} \,|\, \overline{y})$, so function h does have not the metric regularity property.

Definition 3.3. ([4]) A mapping $F : X \rightrightarrows Y$ is positively homogeneous when $0 \in F(0)$ and $F(\lambda x) \supset \lambda F(x)$ for $\lambda > 0$, or equivalently, when $\operatorname{gph} F$ is a cone in $X \times Y$. It is sublinear when, in addition $F(x + x') \supset F(x) + F(x')$, or equivalently, when $\operatorname{gph} F$ is a convex cone in $X \times Y$. It it obvious that if F is positively homogeneous, then F^{-1} is also homogeneous, and mutual.

Definition 3.4. ([10]) When X is a vector space over K, a mapping $p : X \to \mathbb{R}$ is positively homogeneous if $p(x) = \lambda p(x)$, for all $\lambda > 0$, $x \in X$, and sublinear when it is positively homogeneous and in addition $p(x + y) \leq p(x) + p(y)$.

Definition 3.5. ([3]) For a positively homogeneous mapping $F : X \rightrightarrows Y$ the inner norm, is respectively $\|F\|^- = \sup\limits_{x \in \mathbb{B}_X} \inf\limits_{y \in F(x)} \|y\|$. Inner norm can be applied to F^{-1}, $\|F^{-1}\|^- = \sup\limits_{y \in \mathbb{B}_Y} \inf\limits_{x \in F^{-1}(y)} \|x\|$.

Theorem 3.5. ([3]) *If sublinear mapping $F : X \rightrightarrows Y$ and $(\overline{x}, \overline{y}) \in \operatorname{gph} F$, then $\operatorname{reg} F(\overline{x} \,|\, \overline{y}) \leq \operatorname{reg} F(0 \,|\, 0) = \|F^{-1}\|^-$. So F is metrically regulated everywhere if F is regulated from 0 to 0. In this case $\operatorname{reg} F(0 \,|\, 0) = \inf\{k \in (0, \infty) | F(x + kr\mathbb{B}_X) \supset F(x) + r\mathbb{B}_Y$, for all $x \in X, r > 0\} < \infty$. The least equality holds, if and only if F is a onto.*

Proof. We will use for $\operatorname{reg} F(\overline{x} \,|\, \overline{y})$ the definition given in (2.6). From the positive homogeneous mapping we have $F(k\mathbb{B}_X) \supset \operatorname{int} \mathbb{B}_Y$, if we choose $(\overline{x} \,|\, \overline{y}) = (0 \,|\, 0)$.

On the other hand $F(x + kr\mathbb{B}_X) \supset F(x) + rF(k\mathbb{B}_X) \supset y + \operatorname{int} r\mathbb{B}_Y$, where $(x\,,\, y) \in \operatorname{gph} F$, which means $\operatorname{reg} F(\overline{x} \,|\, \overline{y}) \leq \operatorname{reg} F(0 \,|\, 0)$ where $(\overline{x}\,,\, \overline{y}) \in \operatorname{gph} F$. Since $\operatorname{reg} F(0 \,|\, 0) = \inf\{k \in (0, \infty) | F(k\mathbb{B}_X) \supset \mathbb{B}_Y\}$ and

$$\|F^{-1}\|^- = \sup_{y \in \mathbb{B}_Y} \inf_{x \in F^{-1}(y)} \|x\| = \inf\{k \in (0, \infty) | y \in \mathbb{B}_Y \text{ it results } F^{-1}(y) \cap kB_X \neq \phi\}$$

so $\operatorname{reg} F(0 \,|\, 0) = \|F^{-1}\|^-$. Now applying Theorem 2.3 (Robinson-Ursescu) we deduce that in fact $\operatorname{reg} F(0 \,|\, 0) < \infty$, even more this inequality is equivalent to F's surjectivity. Next we will deduce a property of surjectivity Minkowsky's function on a convex absorbent set. We remind that if X is a vector space over K, $Y \subset X$ is called absorbent if for each $x \in X$ there exists $\alpha > 0$, with $x \in \alpha Y$. □

Definition 3.6. ([10]) If $Y \subset X$, Y is absorbent, the mapping $p_Y : X \to \mathbb{R}$ defined by $p_Y(x) = \inf\{\alpha > 0 | \, x \in \alpha Y\}$, $x \in X$ is called Minkowsky's function.

Proposition 3.1. ([10]) *If $Y \subset X$ is absorbent and convex, then*

a) p_Y is positive and positive homogeneous;
b) $Y \subset \{x \in X \,|\, p_Y(x) \leq 1\}$;
c) $\{x \in X \,|\, p_Y(x) < 1\} \subset Y$;
d) p_Y is sublinear.

Corollary 3.5. *We have $\operatorname{reg} p_Y(\overline{x} \,|\, \overline{y}) \leq \operatorname{reg} p_Y(0 \,|\, 0) = 1$, for $Y = \mathbb{B}_X$, and p_Y is onto.*

Proof. By 2.1 Definition we have

$$d(x, p_Y^{-1}(z)) \leq kd(p_Y(x), z).$$

Consider $U = \mathbb{B}_X$, $V = (-1, 1)$, neighborhoods of O in X and 0 in $\mathbb{R}$. For $x \in U$, $z \in V$ we have $d(x, y) \leq kd(p_Y(x), p_Y(y))$ where $p_Y^{-1}(z) = y$, so $z = p_Y(y)$. By $z \in V$ we deduce that $p_Y(y) \leq 1$ which implies $y \in Y$ due to Proposition 3.1 c). We have

$$\|x - y\| \leq k|p_Y(x) - p_Y(y)| \leq k,$$

as $p_Y(x), p_Y(y) \in [0, 1]$.

In fact that, for $x = O \in \mathbb{B}_X$ and $z = 1$ imply $p_Y^{-1}(1) = y$, $p_Y(y) = 1$, so $y \in Y$ is necessary so that $k \geq 1$, that is $\operatorname{reg} p_Y(0 \,|\, 0) = 1$. Due to Theorem 3.5 p_Y is onto. □

Next for C nonempty, closed and convex, from a real local convex space, we define

$$\sigma_C : X^* \to \mathbb{R} \cup \{+\infty\}, \; \sigma_C(f) = sup\{f(c) | c \in C\}$$

called support function. In the case of $C \subset r\,\mathrm{cl}(B_X)$ we have

$$|\sigma_C(f_2) - \sigma_C(f_1)| \leq r\|f_2 - f_1\| \tag{3.10}$$

for all $f_1, f_2 \in X^*$ is checked, where X^* is the dual space ([9]). The relation (3.10) expresses Lipschitz property with the constant r, so the continuity of σ_C.

Corollary 3.6. *From the metric regularity property, we have $\operatorname{reg} \sigma_C(\overline{f} \,|\, \overline{y}) = \dfrac{1}{r}$, where $(\overline{f}, \overline{y}) \in \operatorname{gph} \sigma_C$, $C \subset r \cdot cl\,(\mathbb{B}_X)$.*

Proof. By the metric regularity property we have

$$d(f, \sigma_C^{-1}(y)) = \inf\left\{\|f - g\| \,|\, g \in \sigma_C^{-1}(y)\right\}.$$

If $\sigma_C(f) = \alpha$, then

$$d(y, \alpha) = d(\sigma_C(g), \sigma_C(f)) = |\sigma_C(f) - \sigma_C(g)| \leq r\|f - g\|.$$

We have $\|f - g\| \leq 1 \, \|f - g\| \leq k \cdot r\|f - g\|$, for $k \geq \dfrac{1}{r}$. Choosing $k \geq \dfrac{1}{r}$ the relation $d(f, \sigma_C^{-1}(y)) \leq k \cdot d(y, \sigma_C(f))$ is verified. □

Remark 3.3. We also use in the theorem statement the concept of local closedness for a set at a point, meaning that there exists neighborhood of the point which has a closed intersection with the set.

For a set-valued mapping $F : X \rightrightarrows Y$ which is regulated on $(\overline{x}, \overline{y}) \in \operatorname{gph} F$, we have a result of its perturbation, given by:

Theorem 3.6. ([3]) *Consider a set-valued mapping $F : X \rightrightarrows Y$, $(\overline{x}\,, \overline{y}) \in \operatorname{gph} F$ and $\operatorname{gph} F$ locally closed, and a mapping $G : X \to Y$. If $\operatorname{reg} F(\overline{x}\,|\,\overline{y}) < K < \infty$ and $\operatorname{lip} G(\overline{x}) < \lambda < K^{-1}$, then*

$$\operatorname{reg}\,(F+G)(\overline{x}\,|\,\overline{y}+G(\overline{x})) < \frac{K}{1-\lambda\cdot K}\,.$$

The following result is also demonstrated.

Theorem 3.7. ([4]) *If $F : X \rightrightarrows Y$ is locally closed on $(\overline{x}\,, \overline{y}) \in \operatorname{gph} F$, then*

$$\inf_{G:X\to Y} \left\{ \operatorname{lip} G(\overline{x}) \middle|\, F+G \text{ is not metrically regulated from } \overline{x} \text{ on } \overline{y}+G(\overline{x}) \right\} \geq$$

$$\geq \frac{1}{\operatorname{reg} F(\overline{x}\,|\,\overline{y})}.$$

Definition 3.7. ([8]) If $g : X \to Y$ recall that g is strict differentiable at $\overline{x} \in \operatorname{int}\operatorname{dom} g$ with a strict derivative mapping $Dg(\overline{x}) \in L(X,Y)$ if $\operatorname{lip}\,(g - Dg(\overline{x}))(\overline{x}) = 0$.

Proposition 3.2. ([4]) *If $F(x) = f(x) + K$, where $f : X \to Y$ is strict differentiable on $\overline{x}$, and $K \subset Y$ is a closed convex cone and $\overline{y} \in F(\overline{x})$, $\overline{y} = f(\overline{x})$, then*

$$\inf_{\substack{G:X\to Y\\ G(\overline{x})=0}} \left\{ \operatorname{lip} G(\overline{x}) \,\middle|\, F+G \text{ is not metrically regulated from } \overline{x} \text{ on } \overline{y} \right\} =$$

$$= \frac{1}{\operatorname{reg} F(\overline{x}\,|\,\overline{y})}\,.$$

Definition 3.8. ([4]) If $F_0 : X \rightrightarrows Y$ and $F = F_0 + G$, $G : X \to Y$ with $G(\overline{x}) = 0 = \operatorname{lip} G(\overline{x})$, then F_0 is a strict approximation of the first order of F on $\overline{x}$.

In some cases it involves F_0's metric regularity.

Proposition 3.3. ([4]) *If $F_0 : X \rightrightarrows Y$ is a strict approximation of the first order of $F : X \rightrightarrows Y$ on $\overline{x}$, and $\operatorname{gph} F$ is locally closed on $(\overline{x}\,|\,\overline{y}) \in \operatorname{gph} F$, then F is metrically regulated from $\overline{x}$ on $\overline{y}$ if and only if F_0 is regulated at $\overline{x}$ to $\overline{y}$, and $\operatorname{reg} F(\overline{x}\,|\,\overline{y}) = \operatorname{reg} F_0(\overline{x}\,|\,\overline{y})$.*

Remark 3.4. We consider the cases in which $F = f + M$, where $f : X \to Y$ is continuous, and $M : X \rightrightarrows Y$ has a closed graph and $\overline{y} \in F(\overline{x})$. If f is strict differentiable on $\overline{x}$, then $\operatorname{reg} F(\overline{x}\,|\,\overline{y}) = \operatorname{reg}\,(f_0 + M)(\overline{x}\,|\,\overline{y})$, where $f_0(x) = f(\overline{x}) + Df(\overline{x})(x - \overline{x})$.

In particular case, when $F = f$, we have that $F_0(x) = F(\overline{x}) + DF(\overline{x})(x - \overline{x})$ is an approximation of the first order for F on $\overline{x}$, and $\operatorname{reg} F(\overline{x}\,|\,\overline{y}) = \operatorname{reg} DF(\overline{x}\,|\,\overline{y})$.

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