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On the algebraically compact abelian Q-groups III

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ABSTRACT.

Let κ be a cardinal such that $\kappa < \kappa^{\aleph_0}$. Then we prove here that every algebraically compact abelian Q-group with torsion part of cardinality κ is bounded. This supplies recent results of ours in (Compt. rend. Acad. bulg. Sci., 2007)

1. INTRODUCTION

Throughout this brief note, all groups into consideration are assumed to be additively written Abelian groups as is the custom when dealing with such a theory. All other notion and notation are standard and the same as those in [5] and [7]. For instance, for such a group G, G^1 denotes its first Ulm subgroup consisting of all elements of infinite height in G and, for any set S, $|S|$ denotes its cardinality.

For completeness of the exposition and for making the paper more friendly to the reader, we recollect once again the basic terminology. (See [2], [3] and [4] for more account too.)

Definition 1.1. ([5]). A group G is called algebraically compact if G separates as a direct summand of each group which contains it as a pure subgroup.

Definition 1.2. ([6]). A group G is said to be a Q-group if $G^1 = 0$ and for each $H \leq G$ with $|H| \geq \aleph_0$ the inequality $|(G/H)^1| \leq |H|$ holds.

Definition 1.3. ([6]). A group G is called Fuchs 5 if every infinite subgroup H of G can be embedded in a direct summand K of G such that $|K| = |H|$.

It is not hard to see that any Fuchs 5 group without elements of infinite heights is a Q-group (see, e.g., [6]); nevertheless the converse fails even for cardinality $\aleph_1$.

In [4] we proved that any Fuchs 5 algebraically compact group is a direct sum of a divisible group and a bounded group. Since as aforementioned Fuchs 5 groups with no element of infinite height are themselves Q-groups, we have raised the following conjecture (see also [1], Problem 2).

Conjecture. Algebraically compact Q-groups are bounded.

In a series of recent investigations [2] and [3], we obtained the following two results.

Theorem 1.1. ([2]). *Suppose that G is a group whose torsion subgroup contains only a finite number of primary components (in particular, G is p-mixed for some prime p). Then G is an algebraically compact Q-group if and only if G is bounded.*

Theorem 1.2. ([2], [3]). *Suppose that G is a group whose maximal torsion subgroup is either bounded or countable. Then G is an algebraically compact Q-group if and only if it is bounded.*

The purpose of the present short article is to extend the second half-part of latter Theorem 2.1. in the countable case to groups whose maximal torsion subgroup is with special cardinality κ. Specifically, this cardinality κ satisfies the inequality $\kappa < \kappa^{\aleph_0}$, where $\aleph_0$ is the first countably infinite cardinal. The existence of these cardinals κ is guaranteed via [7]. For instance, observe that $\kappa = \aleph_0$ works, namely $\aleph_0 < \aleph_0^{\aleph_0}$. It is easily seen with the aid of [7] that even $\kappa \neq \mathbf{m}^{\aleph_0}$ for any cardinal $\mathbf{m}$. To be more concrete, $\kappa = \aleph_1$ with $2^{\aleph_0} > \aleph_1$, i.e., the negation of the Continuum Hypothesis ($\neg$ CH) sustained.

So, we are now prepared to proceed by proving the following chief assertion.

2. THE MAIN RESULT

Theorem 2.3. *Each algebraically compact Q-group whose maximal torsion subgroup is of cardinality $\kappa < \kappa^{\aleph_0}$ is bounded.*

Proof. Assume that G is such a group with maximal torsion subgroup tG such that $|tG| = \kappa$. Since G is reduced algebraically compact, whence it does not possess elements of infinite height, according to ([5], Proposition 40.1), we write

$$G = \prod_p G_{(p)}$$

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where $G_{(p)}$ are p-local algebraically compact subgroups of G. But subgroups of Q-groups are again Q-groups (see [6]). Thus, the first stated Theorem 1.1 from [2] applies to show that $G_{(p)}$ are bounded p-groups. Therefore, $\oplus_p G_{(p)}$ is a torsion direct sum of cyclic groups. Observe that

$$tG = \overline{\oplus_p G_{(p)}} = \oplus_p \overline{G_{(p)}} \supseteq \oplus_p G_{(p)}$$

is torsion-complete with a basic subgroup $\oplus_p G_p$. Likewise, $G/\oplus_p G_{(p)}$ and $tG/\oplus_p G_{(p)}$ are known to be divisible (see, for instance, [5]). First of all, let $\oplus_p G_{(p)}$ be infinite. Since G is a Q-group, we deduce that

$$|G/\oplus_p G_{(p)}| \leq |\oplus_p G_{(p)}|$$

hence

$$|G| = |\prod_p G_{(p)}| = |\oplus_p G_{(p)}|$$

and

$$|tG| = |t(\prod_p G_{(p)})| = |\oplus_p G_{(p)}|$$

thus $|G| = |tG|$.

Next, owing to a slight modification of ([5], v. II, p. 29, Exercise 7), we infer that

$$|tG| = |\oplus_p G_{(p)}| = |\oplus_p G_{(p)}|^{\aleph_0} = |tG|^{\aleph_0}$$

provided that tG is unbounded. But this contradicts our assumption on the cardinality of tG and thereby tG must be bounded. If now $\oplus_p G_{(p)}$ is finite, then $tG = \oplus_p G_{(p)}$ by [5] and hence it is again bounded. Consequently, in both cases, the first part-half of the second stated Theorem 1.2 listed above is applicable to conclude that G is, indeed, bounded. In fact, in view of ([5], Theorem 27.5), one may write that $G \cong (tG) \oplus (G/tG)$. Since a direct summand of an algebraically compact Q-group is again an algebraically compact Q-group, we have that G/tG retains the same property. Utilizing ([5], Corollary 40.4), if $G \neq tG$, G/tG possesses a direct summand isomorphic to the group J_p of all p-adic integers for some prime p. Thus, J_p is a Q-group. But $J_p/\mathbf{Z}$ is always divisible (see, for example, [5], v. I, p. 119, Exercise 4 or [3]), where **Z** is the additive groups of all integers. Hence,

$$|J_p/\mathbf{Z}| \leq |\mathbf{Z}|$$

which is tantamount to the equality

$$|J_p| = |\mathbf{Z}| = \aleph_0.$$

However, this is impossible because $|J_p| = 2^{\aleph_0}$ (see e.g. [5]). This contradiction allows us to get that $G = tG$, that is, G is of necessity torsion. Finally, since G is reduced algebraically compact, we wish apply ([5], Corollary 40.3) to infer that G is bounded, as claimed. □

As immediate consequences we derive the quoted above statement from [3] in the countable case and a new statement in the uncountable case.

Corollary 2.1. ([3]). *Every algebraically compact Q-group with countable maximal torsion subgroup is bounded.*

This assertion can be proved even in a more direct manner by observing that $|G| = |tG|$ which was mentioned in the proof of Theorem. So, G must be also countable whence ([5], v. I, p. 200, Exercise 3(a)) is applicable to conclude that G is bounded, indeed.

Recall once again that ($\neg$ CH) means the denial of the Continuum Hypothesis.

Corollary 2.2. ($\neg$ CH). *Every algebraically compact Q-group with maximal torsion subgroup of cardinality $\aleph_1$ is bounded.*

Proof. Observe that $2^{\aleph_0} > \aleph_1$ secures $\aleph_1 < \aleph_1^{\aleph_0} = 2^{\aleph_0}$ because of the inequalities $2^{\aleph_0} \leq \aleph_1^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$. Hereafter, Theorem 2.1 works to infer the claim. □

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