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About approximation of B-continuous and B-differentiable functions of three variables by GBS operators of Bernstein type

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ABSTRACT.

In this article, using a method from the paper [4], the sequence of GBS operators of Bernstein type for B-continuous and B-differentiable functions of three variables is constructed and some approximation properties of this sequence are established.

1. PRELIMINARIES

In the following, let X, Y and Z be compact real intervals and $D = X \times Y \times Z$. A function $f : D \to \mathbb{R}$ is called a B-continuous function at $(x_0, y_0, z_0) \in D$ iff for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\Delta f[(x, y, z), (x_0, y_0, z_0)]| < \epsilon$$

for any $(x, y, z) \in D$, with $|x - x_0| < \delta$, $|y - y_0| < \delta$ and $|z - z_0| < \delta$. Here

$$\Delta f[(x, y, z), (x_0, y_0, z_0)] = f(x, y, z) - f(x, y, z_0) - f(x, y_0, z) - f(x_0, y, z)+$$
$$+ f(x, y_0, z_0) + f(x_0, y, z_0) + f(x_0, y_0, z) - f(x_0, y_0, z_0)$$

denote a so-called mixed difference of f. A function $f : D \to \mathbb{R}$ is called a B-differentiable function at $(x_0, y_0, z_0) \in D$ iff it exists and if the limit is finite

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{\Delta f[(x, y, z), (x_0, y_0, z_0)]}{(x - x_0)(y - y_0)(z - z_0)}.$$

This limit is named the B-differential of f at the point (x_0, y_0, z_0) and is denoted by $Df(x_0, y_0, z_0)$. The function f is B-continuous (B-differentiable) on D iff f is B-continuous (B-differentiable) at each point $(x_0, y_0, z_0) \in D$. These notions were introduced by K. Bögel in [5], [6] and [7]. The function $f : D \to \mathbb{R}$ is B-bounded on D iff there exists $K > 0$ such that $|\Delta f[(x, y, z), (x_0, y_0, z_0)]| \le K$ for any $(x, y, z), (x_0, y_0, z_0) \in D$. We shall use the function sets

$$B(D) = \{f/f : D \to \mathbb{R}, \text{f bounded on } D\}, \text{with the norm } \|\bullet\|_\infty$$
$$B_b(D) = \{f/f : D \to \mathbb{R}, f \text{ B} - \text{bounded on } D\},$$
$$\text{with the norm} \|f\|_B = \sup_{(x,y,z),(u,v,w)\in D} |\Delta f[(x, y, z), (u, v, w)]|$$
$$C_b(D) = \{f/f : D \to \mathbb{R}, f \text{ B} - \text{continuous on } D\}$$
$$D_b = \{f/f : D \to \mathbb{R}, f \text{ B} - \text{differentiable on } D\}.$$

The following mean-value theorem will be useful in the sequel.

Lemma 1.1. If $f \in D_b([a, b] \times [a', b'] \times [a'', b''])$ then there exists $(\xi, \eta, \zeta) \in (a, b) \times (a', b') \times (a'', b'')$ such that

$$\Delta f[(a, a', a''), (b, b', b'')] = (b - a)(b' - a')(b'' - a'')Df(\xi, \eta, \zeta).$$

Let $f \in B_b(D)$. The function

$$\omega_{mixed}(f; \bullet, \bullet, \bullet) : [0, \infty) \times [0, \infty) \times [0, \infty) \to \mathbb{R}$$

defined by

$$\omega_{mixed}(f; \delta_1, \delta_2, \delta_3) = \sup\{|\Delta f[(x, y, z), (x_0, y_0, z_0)]| : |x - x_0| \le \delta_1,$$
$$|y - y_0| \le \delta_2, |z - z_0| \le \delta_3\}$$

for any $\delta_1, \delta_2, \delta_3 \in [0, \infty)$ is called the mixed modulus of smoothness and was introduced by I. Badea in [1] for functions of two variables. Important properties of ω_{mixed} were established by C. Badea, I. Badea, C. Cottin and H.H. Gonska in the papers [2] and [3].

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Lemma 1.2. *Let $f \in B_b(D)$. Then*

$$\omega_{mixed}(f;\delta_1,\delta_2,\delta_3) \le \omega_{mixed}(f;\delta_1',\delta_2',\delta_3') \tag{1.1}$$

for any $\delta_1,\delta_2,\delta_3,\delta_1',\delta_2',\delta_3' \in [0,\infty)$ such that $\delta_1 \le \delta_1'$, $\delta_2 \le \delta_2'$ and $\delta_3 \le \delta_3'$;

$$\Delta f[(x,y,z),(u,v,w)] \le \omega_{mixed}(f;|x-u|,|y-v|,|z-w|); \tag{1.2}$$

$$\Delta f[(x,y,z),(u,v,w)] \le \left(1+\frac{|x-u|}{\delta_1}\right)\left(1+\frac{|y-v|}{\delta_2}\right)\cdot \\ \cdot\left(1+\frac{|z-w|}{\delta_3}\right)\omega_{mixed}(f;\delta_1,\delta_2,\delta_3) \tag{1.3}$$

for $\delta_1,\delta_2,\delta_3>0$;

$$\omega_{mixed}(f;\lambda_1\delta_1,\lambda_2\delta_2,\lambda_3\delta_3) \le (1+\lambda_1)(1+\lambda_2)(1+\lambda_3)\cdot \\ \cdot\,\omega_{mixed}(f;\delta_1,\delta_2,\delta_3),\ \lambda_1,\lambda_2,\lambda_3>0. \tag{1.4}$$

Lemma 1.3. *Let $f \in C_b(D)$. Then*

$$\lim_{\delta_1,\delta_2,\delta_3\to 0}\omega_{mixed}(f;\delta_1,\delta_2,\delta_3)=0. \tag{1.5}$$

Let $B_m : C([0,1]) \to C([0,1])$ be the Bernstein operators, defined for any function $f \in C([0,1])$ and any non-negative integer m by

$$(B_m f)(x)=\sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.6}$$

where $p_{m,k}$ are the fundamental polynomials, defined by

$$p_{m,k}(x)=\binom{m}{k}x^k(1-x)^{m-k}, \tag{1.7}$$

$k \in \{0,1,\dots,m\}$ and $x\in[0,1]$ (see [8]).

Let $e_j:[0,1]\to\mathbb{R}$, $j\in\{0,1,2,3,4\}$, $e_j(x)=x^j$ be the test functions. Then we have (see [9])

Lemma 1.4. *The operators B_m have the properties:*

$$(B_me_0)(x)=1,(B_me_1)(x)=x,\ (B_me_2)(x)=x^2+\frac{x(1-x)}{m},$$
$$(B_me_3)(x)=\frac{(m-1)(m-2)}{m^2}x^3+\frac{3(m-1)}{m^2}x^2+\frac{1}{m^2}x$$

and

$$(B_me_4)(x)=\frac{(m-1)(m-2)(m-3)}{m^3}x^4+\frac{6(m-1)(m-2)}{m^3}x^2+ \\ +\frac{7(m-1)}{m^3}x^2+\frac{1}{m^3}x.$$

2. Introduction

Let L be a positive operator of three variables, applying the space $\mathbb{R}^{[a,b]\times[a',b']\times[a'',b'']}$ into itself. The operator

$$UL:\mathbb{R}^{[a,b]\times[a',b']\times[a'',b'']}\to\mathbb{R}^{[a,b]\times[a',b']\times[a'',b'']}$$

defined by

$$(ULf)(x,y,z)=L[f(\bullet,y,z)+f(x,*,z)+f(x,y,\circ)- \\ -f(\bullet,*,z)-f(\bullet,y,\circ)-f(x,*,\bullet)+f(\bullet,*,\circ);x,y,z] \tag{2.8}$$

is called GBS (generalized boolean sum) operator associated to L, where $\bullet,*,\circ$ stand for the first, the second and the third variable. The term of GBS operator was introduced by C. Badea and C. Cottin in the paper [2].

For B-continuous functions we have the estimation

Theorem 2.1. *For any $f \in C_b(D)$ and any $\delta_1, \delta_2, \delta_3 > 0$ holds the inequality*

$$|(ULf)(x,y,z) - f(x,y,z)| \leq \tag{2.9}$$
$$\leq \left(1 + \frac{\sqrt{(L(\bullet - x)^2)(x,y,z)}}{\delta_1} + \frac{\sqrt{(L(* - y)^2)(x,y,z)}}{\delta_2} + \right.$$
$$+ \frac{\sqrt{(L(\circ - z)^2)(x,y,z)}}{\delta_3} + \frac{\sqrt{(L(\bullet - x)^2(* - y)^2)(x,y,z)}}{\delta_1\delta_2} +$$
$$+ \frac{\sqrt{(L(* - y)^2(\circ - z)^2)(x,y,z)}}{\delta_2\delta_3} + \frac{\sqrt{(L(\circ - z)^2(\bullet - x)^2)(x,y,z)}}{\delta_3\delta_1} +$$
$$\left. + \frac{\sqrt{(L(\bullet - x)^2(* - y)^2)(\circ - z)^2(x,y,z)}}{\delta_1\delta_2\delta_3}\right) \cdot \omega_{mixed}(f;\delta_1,\delta_2,\delta_3)$$

where $L : B(D) \to B(D)$ is a positive linear operator which reproduces the constants.

Proof. After (1.3), for $(x,y,z), (u,v,w) \in D$ we have

$$\Delta f[(x,y,z),(u,v,w)] \leq \left(1 + \frac{|x-u|}{\delta_1}\right) \cdot \left(1 + \frac{|y-v|}{\delta_2}\right) \cdot$$
$$\cdot \left(1 + \frac{|z-w|}{\delta_3}\right) \omega_{mixed}(f;\delta_1,\delta_2,\delta_3),$$

so that we can write

$$|f(x,y,z) - (ULf)(x,y,z)| \leq |(L\Delta f[(x,y,z)(\bullet,*,\circ)])(x,y,z)| \leq$$
$$\leq \left(1 + \frac{(L|\bullet - x|)(x,y,z)}{\delta_1} + \frac{(L|* - y|)(x,y,z)}{\delta_2} + \frac{(L|\circ - z|)(x,y,z)}{\delta_3} + \right.$$
$$+ \frac{(L|\bullet - x||* - y|)(x,y,z)}{\delta_1\delta_2} + \frac{(L|* - y||\circ - z|)(x,y,z)}{\delta_2\delta_3} +$$
$$\left. + \frac{(L|\circ - z||\bullet - x|)(x,y,z)}{\delta_3\delta_1} + \frac{(L|\bullet - x||* - y||\circ - z|)(x,y,z)}{\delta_1\delta_2\delta_3}\right) \cdot$$
$$\cdot\, \omega_{mixed}(f,\delta_1,\delta_2,\delta_3).$$

Applying the Cauchy-Schwarz inequality for positive linear operators the estimation from theorem results. □

For B-differentiable functions we have the estimation

Theorem 2.2. *Let $L : C_b(D) \to B(D)$ be a positive linear operator which reproduces the constants and $UL : C_b(D) \to B(D)$ the associated GBS operator. Then for any function $f \in D_b(D)$ with $Df \in B(D)$ and any $\delta_1, \delta_2, \delta_3 > 0$ we have*

$$|(ULf)(x,y,z) - f(x,y,z)| \leq \tag{2.10}$$
$$\leq 7\|Df\|_\infty \sqrt{(L(\bullet - x)^2(* - y)^2(\circ - z)^2)(x,y,z)} +$$
$$+ [\sqrt{(L(\bullet - x)^2(* - y)^2(\circ - z)^2)(x,y,z)} +$$
$$+ \delta_1^{-1}\sqrt{(L(\bullet - x)^4(* - y)^2(\circ - z)^2)(x,y,z)} +$$
$$+ \delta_2^{-1}\sqrt{(L(\bullet - x)^2(* - y)^4(\circ - z)^2)(x,y,z)} +$$
$$+ \delta_3^{-1}\sqrt{(L(\bullet - x)^2(* - y)^2(\circ - z)^4)(x,y,z)} +$$
$$+ \delta_1^{-1}\delta_2^{-1}\sqrt{(L(\bullet - x)^4(* - y)^4(\circ - z)^2)(x,y,z)} +$$
$$+ \delta_2^{-1}\delta_3^{-1}\sqrt{(L(\bullet - x)^2(* - y)^4(\circ - z)^4)(x,y,z)} +$$
$$+ \delta_3^{-1}\delta_1^{-1}\sqrt{(L(\bullet - x)^4(* - y)^2(\circ - z)^4)(x,y,z)} +$$
$$+ \delta_1^{-1}\delta_2^{-1}\delta_3^{-1}\sqrt{(L(\bullet - x)^4(* - y)^4(\circ - z)^4)(x,y,z)}] \cdot$$
$$\cdot \omega_{mixed}(Df;\delta_1,\delta_2,\delta_3).$$

Proof. For $(x,y,z), (u,v,w) \in D$ we can write

$$|(ULf)(x,y,z) - f(x,y,z)| \leq |L\Delta f[(x,y,z),(u,v,w)](x,y,z)|$$

and after Lemma 1.1 we have

$$\Delta f[(x,y,z),(u,v,w)] = (u-x)(v-y)(w-z)Df(\xi,\eta,\zeta) =$$
$$= (u-x)(v-y)(w-z)(-\Delta Df[(x,y,z),(\xi,\eta,\zeta)] + Df(x,y,z) -$$
$$- Df(\xi,y,z) - Df(x,\eta,z) - Df(x,y,\zeta) + Df(x,\eta,\zeta) +$$
$$+ Df(\xi,y,\zeta) + Df(\xi,\eta,z))$$

so that we have

$$\begin{gathered}
|(ULf)(x,y,z)-f(x,y,z)| \leq (L|u-x||v-y||w-z|\cdot \\
\cdot|\Delta Df[(x,y,z),(\xi,\eta,\zeta)]|)(x,y,z)+ \\
+(L|u-x||v-y||w-z|(|Df(x,y,z)|+|Df(\xi,y,z)|+|Df(x,\eta,z)|+ \\
+|Df(x,y,\zeta)|+|Df(x,\eta,\zeta)|+|Df(\xi,y,\zeta)|+|Df(\xi,\eta,z)|))(x,y,z) \leq \\
\leq 7\|Df\|_\infty (L|u-x||v-y||w-z|)(x,y,z)+ \\
+(L|u-x||v-y||w-z|\omega_{mixed}(Df;|\xi-x|,|\eta-y|,|\zeta-z|))(x,y,z) \leq \\
\leq 7\|Df\|_\infty \left(L|u-x||v-y||w-z|\right)(x,y,z)+ \\
+\left(L|u-x||v-y||w-z|\left(1+\frac{|u-x|}{\delta_1}\right)\left(1+\frac{|v-y|}{\delta_2}\right)\cdot\right. \\
\left.\cdot\left(1+\frac{|w-z|}{\delta_3}\right)\right)(x,y,z)\omega_{mixed}(Df;\delta_1,\delta_2,\delta_3) = \\
=7\|Df\|_\infty (L|u-x||v-y||w-z|)(x,y,z)+ \\
+L\left(|u-x||v-y||w-z|+\frac{|u-x|^2|v-y||w-z|}{\delta_1}+\right. \\
+\frac{|u-x||v-y|^2|w-z|}{\delta_2}+\frac{|u-x||v-y||w-z|^2}{\delta_3}+ \\
+\frac{|u-x|^2|v-y|^2|w-z|}{\delta_1\delta_2}+\frac{|u-x||v-y|^2|w-z|^2}{\delta_2\delta_3}+ \\
\left.+\frac{|u-x|^2|v-y||w-z|^2}{\delta_3\delta_1}+\frac{|u-x|^2|v-y|^2|w-z|^2}{\delta_1\delta_2\delta_3}\right)(x,y,z)\cdot \\
\cdot\omega_{mixed}(Df;\delta_1,\delta_2,\delta_3).
\end{gathered}$$

Applying the Cauchy-Schwarz inequality for positive linear operators the estimation from theorem results. □

Let $B_{l,m,n}: C([0,1]^3) \to C([0,1]^3)$, l,m,n non-negative integers, be the Bernstein operators for functions of three variables, defined by

$$(B_{l,m,n}f)(x,y,z)=\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} p_{l,i}(x)p_{m,j}(y)p_{n,k}(z)\cdot f\left(\frac{i}{l},\frac{j}{m},\frac{k}{n}\right), \tag{2.11}$$

$f \in C([0,1]^3)$, and let B_l^x, B_m^y and B_n^z the parametric extensions of the operators (1.6) defined for any function $f \in C([0,1]^3)$ by

$$\begin{aligned}
(B_l^x f)(x,y,z) &= \sum_{i=0}^{l} p_{l,i}(x) f\left(\frac{i}{l},y,z\right) = (B_{l,m,n}f(\bullet,y,z))(x,y,z) \\
(B_m^y f)(x,y,z) &= \sum_{j=0}^{m} p_{m,j}(y) f\left(x,\frac{j}{m},z\right) = (B_{l,m,n}f(x,*,z))(x,y,z) \\
(B_n^z f)(x,y,z) &= \sum_{k=0}^{n} p_{n,k}(x) f\left(x,y,\frac{k}{n}\right) = (B_{l,m,n}f(x,y,\circ))(x,y,z).
\end{aligned}$$

The GBS operator of Bernstein type is the boolean sum

$$UB_{l,m,n}=B_l^x+B_m^y+B_n^z-B_l^xB_m^y-B_m^yB_n^z-B_n^zB_l^x+B_l^xB_m^yB_n^z$$

where

$$\begin{aligned}
&(B_l^xB_m^y f)(x,y,z)=\sum_{i=0}^{l}\sum_{j=0}^{m} p_{l,i}(x)p_{m,j}(y) f\left(\frac{i}{l},\frac{j}{m},z\right)= \\
&=(B_{l,m,n}f(\bullet,*,z))(x,y,z) \\
&(B_m^yB_n^z f)(x,y,z)=\sum_{j=0}^{m}\sum_{k=0}^{n} p_{m,j}(y)p_{n,k}(z) f\left(x,\frac{j}{m},\frac{k}{n}\right)= \\
&=(B_{l,m,n}f(x,*,\circ))(x,y,z) \\
&(B_n^zB_l^x f)(x,y,z)=\sum_{k=0}^{n}\sum_{i=0}^{l} p_{n,k}(z)p_{l,i}(x) f\left(\frac{i}{l},y,\frac{k}{n}\right)= \\
&=(B_{l,m,n}f(\bullet,y,\circ))(x,y,z) \\
&(B_l^xB_m^yB_n^z f)(x,y,z)=(B_{l,m,n}f)(x,y,z).
\end{aligned}$$

3. Main results

About approximation for B-continuous functions of three variables by Bernstein type operators, we have

Theorem 3.3. *For any $f \in C_b([0,1]^3)$ and any $(x,y,z) \in [0,1]^3$ we have the inequality*

$$|(UB_{l,m,n}f)(x,y,z) - f(x,y,z)| \leq 8\omega_{mixed}\left(f; \frac{1}{2\sqrt{l}}, \frac{1}{2\sqrt{m}}, \frac{1}{2\sqrt{n}}\right). \tag{3.12}$$

Proof. We have

$$(B_{l,m,n}(\bullet - x)^2)(x,y,z) = (B_{l,m,n}e_{200})(x,y,z) -$$
$$-2x(B_{l,m,n}e_{100})(x,y,z) + x^2(B_{l,m,n}e_{000})(x,y,z) = \frac{x(1-x)}{l} \leq \frac{1}{4l},$$

where $e_{ijk} : [0,1]^3 \to \mathbb{R}$, $e_{ijk}(x,y,z) = x^i y^j z^k$, $i,j,k \in \{0,1,2,3,4\}$ are the test functions. In a similar way can be established the following

$$(B_{l,m,n}(* - y)^2)(x,y,z) = \frac{y(1-y)}{m} \leq \frac{1}{4m}$$
$$(B_{l,m,n}(\circ - z)^2)(x,y,z) = \frac{z(1-z)}{n} \leq \frac{1}{4n}$$
$$(B_{l,m,n}(\bullet - x)^2(* - y)^2)(x,y,z) = \frac{xy(1-x)(1-y)}{lm} \leq \frac{1}{16lm}$$
$$(B_{l,m,n}(* - y)^2(\circ - z)^2)(x,y,z) = \frac{yz(1-y)(1-z)}{mn} \leq \frac{1}{16mn}$$
$$(B_{l,m,n}(\circ - z)^2(\bullet - x)^2)(x,y,z) = \frac{zx(1-z)(1-x)}{nl} \leq \frac{1}{16nl}$$
$$(B_{l,m,n}(\bullet - x)^2(* - y)^2)(\circ - z)^2(x,y,z) = \frac{xyz(1-x)(1-y)(1-z)}{lmn} \leq$$
$$\leq \frac{1}{64lmn}.$$

Choosing by $\delta_1 = \dfrac{1}{2\sqrt{l}}$, $\delta_2 = \dfrac{1}{2\sqrt{m}}$ and $\delta_3 = \dfrac{1}{2\sqrt{n}}$ in Theorem 2.1 we obtain the estimation (3.1). □

Corollary 3.1. *If $f \in C_b([0,1]^3)$ then the sequence $(UB_{l,m,n})_{l,m,n\in\mathbb{N}}$ converges to f uniformly on $[0,1]^3$.*

For B-differentiable functions we have the estimation:

Theorem 3.4. *For any $f \in D_b([0,1]^3)$ such that $Df \in B([0,1]^3)$, any $(x,y,z) \in [0,1]^3$ and any $\delta_1, \delta_2, \delta_3 > 0$ we have*

$$|(UB_{l,m,n}f)(x,y,z) - f(x,y,z)| \leq \frac{7}{8\sqrt{lmn}}\|Df\|_\infty + \frac{1}{8\sqrt{lmn}} \cdot \tag{3.13}$$
$$\cdot \left(1 + \frac{1}{\delta_1\sqrt{l}}\right)\left(1 + \frac{1}{\delta_2\sqrt{m}}\right)\left(1 + \frac{1}{\delta_3\sqrt{n}}\right)\omega_{mixed}(Df; \delta_1, \delta_2, \delta_3)$$

and

$$|(UB_{l,m,n}f)(x,y,z) - f(x,y,z)| \leq \frac{7}{8\sqrt{lmn}}\|Df\|_\infty + \tag{3.14}$$
$$+ \frac{1}{\sqrt{lmn}}\omega_{mixed}\left(Df; \frac{1}{\sqrt{l}}, \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right),$$

l, m, n positive integers, $l, m, n \geq 2$.

Proof. We have

$$(B_{l,m,n}(\bullet - x)^2(* - y)^2(\circ - z)^2)(x,y,z) = \frac{xyz(1-x)(1-y)(1-z)}{lmn} \leq$$
$$\leq \frac{1}{64lmn};$$
$$(B_{l,m,n}(\bullet - x)^4(* - y)^2(\circ - z)^2)(x,y,z) = (B_{l,m,n}(\bullet - x)^4)(x,y,z) \cdot$$
$$\cdot (B_{l,m,n}(* - y)^2)(x,y,z)(B_{l,m,n}(\circ - z)^2)(x,y,z) \leq \frac{1}{64l^2mn}$$

because

$$\begin{aligned}&(B_{l,m,n}(\bullet - x)^4)(x,y,z) = (B_{l,m,n}e_{400})(x,y,z) - 4x(B_{l,m,n}e_{300})(x,y,z)+\\&+6x^2(B_{l,m,n}e_{200})(x,y,z) - 4x^3(B_{l,m,n}e_{100})(x,y,z)+\\&+x^4(B_{l,m,n}e_{000})(x,y,z) = \frac{(l-1)(l-2)(l-3)}{l^3}x^4 + \frac{6(l-1)(l-2)}{l^3}x^3+\\&+\frac{7(l-1)}{l^3}x^2 + \frac{1}{l^3}x - 4x\left(\frac{(l-1)(l-2)}{l^2}x^3 + \frac{3(l-1)}{l^2}x^2 + \frac{1}{l^2}x\right)+\\&+6x^2\left(\frac{l-1}{l}x^2 + \frac{1}{l}x\right) - 4x^4 + x^4 = \frac{3(l-2)x^2(1-x)^2}{l^3}+\\&+\frac{x(1-x)}{l^3} \le \frac{3l-2}{16l^3} < \frac{1}{4l^2}.\end{aligned}$$

In a similar way can be established the following estimations:

$$\begin{aligned}(B_{l,m,n}(\bullet - x)^2(* - y)^4(\circ - z)^2)(x,y,z) &\le \frac{1}{64lm^2n};\\(B_{l,m,n}(\bullet - x)^2(* - y)^2(\circ - z)^4)(x,y,z) &\le \frac{1}{64lmn^2};\\(B_{l,m,n}(\bullet - x)^4(* - y)^4(\circ - z)^2)(x,y,z) &\le \frac{1}{64l^2m^2n};\\(B_{l,m,n}(\bullet - x)^4(* - y)^2(\circ - z)^4)(x,y,z) &\le \frac{1}{64l^2mn^2};\\(B_{l,m,n}(\bullet - x)^2(* - y)^4(\circ - z)^4)(x,y,z) &\le \frac{1}{64lm^2n^2}\end{aligned}$$

so that the estimation (3.2) results. Choosing by $\delta_1 = \frac{1}{\sqrt{l}}$, $\delta_2 = \frac{1}{\sqrt{m}}$ and $\delta_3 = \frac{1}{\sqrt{n}}$ in Theorem 2.2 we obtain the inequality (3.3). □

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