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Some convergence results for two new iteration processes in uniformly convex Banach space

M. O. OLATINWO, O. O. OWOJORI AND A. P. AKINOLA

ABSTRACT.

In this paper, following the concepts in [7, 9], we shall establish some convergence results for nonexpansive operators in a uniformly convex Banach space. Two new iteration processes will be considered for this purpose. Our results improve, generalize and extend those of [5, 8, 9, 10, 13, 14].

1. INTRODUCTION

Suppose that $A = (a_{nk})$ is an infinite, lower triangular, regular row-stochastic matrix, E a closed convex subset of a Banach space and T a continuous mapping of E into itself and $x_1 \in E$. Then, the general Mann iteration process $M(x_1, A, T)$ which was introduced in Mann [10] is defined by

$$v_n = \sum_{k=1}^{n} a_{nk}x_k,\ x_{n+1} = Tv_n,\ n = 1, 2, \cdots, \tag{1.1}$$

If A is the identity matrix, then each sequence of $M(x_1, A, T)$ becomes the sequence of Picard iterates of T at x_1. It was established in [10] that if either of the sequences $\{x_n\}$ and $\{v_n\}$ converges, then the other also converges to the same point, and their common limit is a fixed point of T.

In [7, 9], it is said that the matrix A is *segmenting* for the Mann process if $a_{n+1,k} = (1 - a_{n+1,n+1})a_{nk}$ for $k \leq n$. In this case, v_{n+1} lies on the segment joining v_n and Tv_n :

$$v_{n+1} = (1 - d_n)v_n + d_nTv_n,\ n = 1, 2, \cdots, \tag{1.2}$$

where $d_n = a_{n+1,n+1}$.

A segmenting matrix is determined by its sequence of diagonal elements. Some authors including [5, 13, 14] have investigated the case $d_n = \lambda,\ 0 < \lambda < 1$, while Mann [10] approximated the fixed points of continuous functions on a closed interval of the real line using the segmenting matrix determined by $d_n = \frac{1}{n}\ \forall\ n$. Dotson [8] considered the case when d_n is bounded away from 0 and 1. Groetsch [9] generalized the results of [5, 8, 10, 13, 14] in a uniformly convex Banach space by employing (2) and assuming that A is a segmenting matrix for which

$$\sum_{n=1}^{\infty} d_n(1 - d_n) = \infty.$$

We shall give another definition of a segmenting matrix in the next section with a view to generalizing and extending Groetsch [9] and others mentioned earlier in this paper.

2. PRELIMINARIES

We shall introduce and employ the following iteration processes: Let E be a Banach space, $T_i : E \to E$ $(i = 0, 1, \cdots, k)$ selfmaps of E and $x_0 \in E$. Then, define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$\left.\begin{array}{c} x_{n+1} = \alpha_{n,0}x_n + \sum_{i=1}^{k} \alpha_{n,i}T_iy_n, \sum_{i=0}^{k} = 1, \quad n = 0, 1, 2..., \\ y_n = \sum_{j=0}^{s} \beta_{n,j}T_jx_n, \sum_{j=0}^{s} \beta_{n,j} = 1 \end{array}\right\} \tag{1.3}$$

$k \geq s,\ \alpha_{n,\ i} \geq 0,\ \alpha_{n,\ 0} \neq 0,\ \beta_{n,\ j} \geq 0,\ \beta_{n,\ 0} \neq 0,\ \alpha_{n,\ i},\ \beta_{n,\ j} \in [0, 1]$, where k and s are fixed integers and T_0 is an identity operator. If $s = 0$ in (3), we also obtain the following interesting iteration process in a Banach space:

$$x_{n+1} = \sum_{i=0}^{k} \alpha_{n,\ i}\ T_ix_n,\ \sum_{i=0}^{k} \alpha_{n,\ i} = 1,\ n = 0, 1, 2, ..., \tag{1.4}$$

$\alpha_{n,\ i} \geq 0,\ \alpha_{n,\ 0} \neq 0,\ \alpha_{n,\ i} \in [0, 1]$, where k is a fixed integer and T_0 is an identity operator.

(i) If $s = 0,\ k = 1$ in (3), then we have $y_n = \beta_{n,\ 0}\, x_n = x_n,\ \beta_{n,\ 0} = 1$ and

$$x_{n+1} = (1 - \alpha_{n,\ 1})x_n + \alpha_{n,\ 1}T_1x_n,$$

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which is the usual Mann iteration process with

$$\sum_{i=0}^{1} \alpha_{n,\, i} = 1,\ \alpha_{n,\, 1} = \alpha_n.$$

(ii) Also, if $s = k = 1$, in (3), we get

$$\left.\begin{array}{r} x_{n+1} = (1-\alpha_{n,\, 1})x_n + \alpha_{n,\, 1}T_1 y_n \\ y_n = (1-\beta_{n,\, 1})x_n + \beta_{n,\, 1}T_1 x_n, \end{array}\right\}$$

which is the usual Ishikawa iteration process with

$$\sum_{i=0}^{1} \alpha_{n,\, i} = \sum_{i=0}^{1} \beta_{n,\, i} = 1, \alpha_{n,\, 1} = \alpha_n,\ \beta_{n,\, 1} = \beta_n.$$

(iii) If $s = 0,\ \alpha_{n,\, i} = \alpha_i$ and $T_i = T^i$ in (3), then we obtain the usual Kirk's iteration process

$$x_{n+1} = \sum_{i=0}^{k} \alpha_i\, T^i x_n,\ \sum_{i=0}^{k} \alpha_i = 1,\ n = 0, 1, 2, \cdots, \tag{1.5}$$

with $y_n = \beta_{n,\, 0}\, x_n = x_n$, since $\beta_{n,\, 0} = 1$.

Eqn. (4) is also a generalization of Picard, Schaefer, Mann and the Kirk's iteration processes. See Berinde [1, 2] for detail on the various already existing fixed point iteration processes.

In this paper, we shall establish some convergence results for nonexpansive operators in a uniformly convex Banach space using the newly introduced iteration processes defined in (3) and (4). We shall assume that A is a segmenting matrix such that

$$\sum_{n=0}^{\infty} \alpha_{n,\, 0}(1-\alpha_{n,\, 0}) = \infty.$$

Our results improve, generalize and extend those of [5, 8, 9, 10, 13, 14].

Lemma 2.1. *(Groetsch [9]): Let X be a uniformly convex Banach space and let $x,\ y \in X$. If $||x|| \le 1,\ ||y|| \le 1$ and $||x-y|| \ge \epsilon > 0$, then $||\lambda x + (1-\lambda)y|| \le 1 - 2\lambda(1-\lambda)\delta(\epsilon)$ for $0 \le \lambda < 1$.*

3. The Main Results

Theorem 3.1. *Let E be a convex subset of a uniformly convex Banach space X and $T_i : E \to E$ $(i = 0, 1, 2, \cdots, k)$ nonexpansive mappings with at least a common fixed point. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (3). Then, the sequence $\left\{(I - T_i^j)x_n\right\}_{n=0}^{\infty}$, for each $j \in \mathbb{N},\ 1 \le j \le k$, converges to $0 \in E$ for each i such that $\sum_{n=0}^{\infty} \alpha_{n,\, 0}(1-\alpha_{n,\, 0}) = \infty$.*

Proof. If p is a common fixed point of T_i for each i, then

$$\begin{aligned} ||x_{n+1} - p|| &= ||\alpha_{n,\, 0}x_n + \sum_{i=1}^{k} \alpha_{n,\, i}T_i y_n - \sum_{i=0}^{k} \alpha_{n,\, i}T_i p|| \le \\ &\le \alpha_{n,\, 0}||x_n - p|| + \sum_{i=1}^{k} \alpha_{n,\, i}||y_n - p|| = \\ &= \alpha_{n,\, 0}||x_n - p|| + \sum_{i=1}^{k} \alpha_{n,\, i}||\sum_{j=0}^{s} \beta_{n,\, j}T_j x_n - \sum_{j=0}^{s} \beta_{n,\, j}T_j p|| = \\ &\le \alpha_{n,\, 0}||x_n - p|| + \sum_{i=1}^{k} \alpha_{n,\, i}\sum_{j=0}^{s} \beta_{n,\, j}||T_j x_n - T_j p|| \le \\ &\le \alpha_{n,\, 0}||x_n - p|| + \sum_{i=1}^{k} \alpha_{n,\, i}\sum_{j=0}^{s} \beta_{n,\, j}||x_n - p|| = \\ &= \sum_{i=0}^{k} \alpha_{n,\, i}||x_n - p|| = ||x_n - p||. \end{aligned}$$

Now,

$$\begin{aligned} ||(I - T_i^j)x_n|| &= ||x_n - T_i^j x_n|| \le ||x_n - p|| + ||p - T_i^j x_n|| \\ &= ||x_n - p|| + ||T_i^j p - T_i^j x_n|| \le 2||x_n - p||. \end{aligned}$$

Suppose on the contrary that $\{(I - T^j)x_n\}_{n=0}^{\infty}$ does not converge to 0. Since

$$||x_n - T_i^j x_n|| \leq 2||x_n - p||,$$

we may assume that there is an $a > 0,\ a \in (0, 1)$ such that $||x_n - p|| \geq a$, for any n. If $\{(I - T^j)x_n\}_{n=0}^{\infty}$ does not converge to 0, then there is an $\epsilon > 0$ such that

$$||x_n - T_i^j x_n|| \geq \epsilon,\ \forall\ n.$$

Let

$$b = 2\delta\left(\frac{\epsilon}{||x_0 - p||}\right),\ x_n = \frac{x_n - p}{||x_n - p||}$$

and

$$z_n = \frac{\sum_{i=1}^{k} \alpha_{n,\ i}(T_i y_n - T_i p)}{(1 - \alpha_{n,\ 0})||x_n - p||}$$

Then, we have

$$||x_n|| \leq \frac{||x_n - p||}{||x_n - p||} = 1$$

and

$$\begin{aligned} ||z_n|| &\leq \frac{\sum_{i=1}^{k} \alpha_{n,\ i}||T_i y_n - T_i p||}{(1 - \alpha_{n,\ 0})||x_n - p||} \leq \\ &\leq \frac{\sum_{i=1}^{k} \alpha_{n,\ i}||y_n - p||}{(1 - \alpha_{n,\ 0})||x_n - p||} \leq \\ &\leq \frac{\sum_{i=1}^{k} \alpha_{n,\ i} \sum_{j=0}^{s} \beta_{n,\ j}||x_n - p||}{(1 - \alpha_{n,\ 0})||x_n - p||} = \\ &= \frac{\sum_{i=1}^{k} \alpha_{n,\ i}||x_n - p||}{(1 - \alpha_{n,\ 0})||x_n - p||} = 1, \end{aligned}$$

since

$$\sum_{i=1}^{k} \alpha_{n,\ i} = 1 - \alpha_{n,\ 0}.$$

Hence, we have from (3) that

$$\begin{aligned} &||x_{n+1} - p|| = ||\alpha_{n,\ 0}x_n + \textstyle\sum_{i=1}^{k} \alpha_{n,\ i}T_i y_n - \sum_{i=0}^{k} \alpha_{n,\ i}T_i p|| = \\ &= ||(||x_n - p||)[\alpha_{n,\ 0}\tfrac{(x_n - p)}{||x_n - p||} + (1 - \alpha_{n,\ 0})\tfrac{\sum_{i=1}^{k} \alpha_{n,\ i}(T_i y_n - T_i p)}{(1 - \alpha_{n,\ 0})||x_n - p||}]|| \\ &\quad \leq ||x_n - p||\ ||\alpha_{n,\ 0}x_n + (1 - \alpha_{n,\ 0})z_n||. \end{aligned} \tag{3.6}$$

Using Lemma 2.1 in (6) yields

$$\begin{aligned} ||x_{n+1} - p|| &\leq [1 - \alpha_{n,\ 0}(1 - \alpha_{n,\ 0})b]||x_n - p|| \\ &= ||x_n - p|| - b\alpha_{n,\ 0}(1 - \alpha_{n,\ 0})||x_n - p|| \\ &\leq ||x_{n-1} - p|| - b\alpha_{n-1,\ 0}(1 - \alpha_{n-1,\ 0})||x_{n-1} - p|| - b\alpha_{n,\ 0}(1 - \alpha_{n,\ 0})||x_n - p|| \end{aligned}$$

Repeating this process inductively leads to

$$\begin{aligned} a &\leq ||x_{n+1} - p|| \leq \\ &\leq ||x_0 - p|| - b[\alpha_{0,\ 0}(1 - \alpha_{0,\ 0})||x_0 - p|| + \alpha_{1,\ 0}(1 - \alpha_{1,\ 0})||x_1 - p|| + \cdots \\ &\quad ... + \alpha_{n,\ 0}(1 - \alpha_{n,\ 0})||x_n - p||\ ] \\ &= ||x_0 - p|| - b\sum_{r=0}^{n} \alpha_{r,\ 0}(1 - \alpha_{r,\ 0})||x_r - p|| \\ &\leq ||x_0 - p|| - ab\sum_{r=0}^{n} \alpha_{r,\ 0}(1 - \alpha_{r,\ 0}) \end{aligned}$$

Therefore, we obtain

$$a[1 + b\sum_{r=0}^{n} \alpha_{r,\ 0}(1 - \alpha_{r,\ 0})] \leq ||x_0 - p||,$$

from which it follows that

$$a \leq \frac{||x_0 - p||}{1 + b\sum_{r=0}^{n} \alpha_{r,\ 0}(1 - \alpha_{r,\ 0})} \to 0 \text{ as } n \to \infty,$$

leading to a contradiction. Therefore, we have $a = 0$. Hence,

$$\lim_{n\to\infty} ||x_n - T_i^j x_n|| = 0,$$

for each i. □

Remark 3.1. *Theorem 3.1 is a generalization of the results of* [5, 8, 9, 10, 13, 14].

Theorem 3.2. *Let E be a convex subset of a uniformly convex Banach space X and $T_i : E \to E$ $(i = 0, 1, 2, \cdots, k)$ nonexpansive mappings with at least a common fixed point. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (4). Then, the sequence $\left\{(I - T_i^j)x_n\right\}_{n=0}^{\infty}$, for each $j \in \mathbb{N},\ 1 \le j \le k$, converges to $0 \in E$ for each i such that*

$$\sum_{n=0}^{\infty} \alpha_{n,\,0}(1 - \alpha_{n,\,0}) = \infty.$$

Proof. The proof of this theorem is similar to that of Theorem 3.1. □

Remark 3.2. *Theorem 3.3 is also a generalization of the results of* [5, 8, 9, 10, 13, 14].

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Department of Mathematics
Obafemi Awolowo University
Ile-Ife, Nigeria
E-mail address: polatinwo@oauife.edu.ng
E-mail address: molaposi@yahoo.com
E-mail address: walejori@oauife.edu.ng
E-mail address: aakinola@oauife.edu.ng