New category of inequalities

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1. Introduction

In the following, we consider the sets
\[ D_1(a) = \{ x \in \mathbb{R} : x \geq a \}, \quad S_1(a) = \{ x \in \mathbb{R} : x \leq a \}, \]
\[ D_2(a, b) = \{ (x, y) \in \mathbb{R}^2 : ax + y \geq b \}, \quad S_2(a, b) = \{ (x, y) \in \mathbb{R}^2 : ax + y \leq b \}, \]
where \( a, b \in \mathbb{R} \) and let \( N = \{1, 2, 3, \ldots \} \).
These offer light conditions for some classical inequalities.

2. Inequalities in \( D_1 \) and \( S_1 \)

**Theorem 2.1.** If \( n \in \mathbb{N}, x_k \in D_1(0), k \in \{1, 2, \ldots, n\} \) and \( \alpha \geq 1 \), then
\[
\sum_{k=1}^{n} \frac{x_k}{1 + (\alpha - 1)x_k} \leq \frac{n^2}{(n + \alpha - 1)^2} \sum_{k=1}^{n} x_k + \frac{n(\alpha - 1)}{(n + \alpha - 1)^2}.
\]

**Proof.** If \( x > 0 \) and \( \alpha \geq 1 \), the inequality \( \frac{x}{1 + (\alpha - 1)x} \leq \frac{n^2 x + \alpha - 1}{(n + \alpha - 1)^2} \) holds, because it is equivalent with \( (\alpha - 1)(nx - 1)^2 \geq 0 \). For \( x \in \{x_1, x_2, \ldots, x_n\} \) and summing the inequalities obtained, we have that
\[
\sum_{k=1}^{n} \frac{x_k}{1 + (\alpha - 1)x_k} \leq \sum_{k=1}^{n} \frac{n^2 x_k + \alpha - 1}{(n + \alpha - 1)^2},
\]
from where (2.1) results.

**Corollary 2.1.** If \( n \in \mathbb{N}, \alpha \geq 1, y_k \in D_1(0), k \in \{1, 2, \ldots, n\} \) and \( y_1 + y_2 + \cdots + y_n \neq 0 \), then
\[
\sum_{cyclic} \frac{y_1}{y_1 + y_2 + \cdots + y_n} \leq \frac{n}{n + \alpha - 1}.
\]

**Proof.** In Theorem 2.1 we take \( x_k = \frac{y_k}{y_1 + y_2 + \cdots + y_n}, k \in \{1, 2, \ldots, n\} \).

**Theorem 2.2.** If \( n \in \mathbb{N}\setminus\{2\} \) and \( x_k \in S_1\left(\frac{1}{2}\right) \setminus \left\{\frac{1}{2}\right\}, k \in \{1, 2, \ldots, n\} \), then
\[
\frac{2n}{(n - 2)^2} + \sum_{k=1}^{n} \frac{x_k}{1 - 2x_k} \geq \left(\frac{n}{n - 2}\right)^2 \sum_{k=1}^{n} x_k.
\]

**Proof.** If \( x < \frac{1}{2} \) and \( n \in \mathbb{N}\setminus\{2\} \), the inequality \( \frac{x}{1 - 2x} \geq \frac{n^2 x - 2}{(n - 2)^2} \) is true because it is equivalent with \( (nx - 1)^2 \geq 0 \).

Taking the inequality above account, we have that \( \sum_{k=1}^{n} \frac{x_k}{1 - 2x_k} \geq \sum_{k=1}^{n} \frac{n^2 x_k - 2}{(n - 2)^2} \), from where (2.3) results.

**Corollary 2.2.** Let \( n \in \mathbb{N}, n \geq 3 \). In all convex polygons with sides \( y_1, y_2, \ldots, y_n \), the inequality
\[
\sum_{cyclic} \frac{y_1}{y_2 + y_3 + \cdots + y_n - y_1} \geq \frac{n}{n - 2}
\]
holds.

**Proof.** In Theorem 2.2 we take \( x_k = \frac{y_k}{y_1 + y_2 + \cdots + y_n}, k \in \{1, 2, \ldots, n\} \) and it is verified immediately that \( x_k < \frac{1}{2}, k \in \{1, 2, \ldots, n\} \) because \( y_1, y_2, \ldots, y_n \) are the sides of a convex polygon.

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**Theorem 2.3.** If \( n \in \mathbb{N} \setminus \{1\} \) and \( x_k \in S_1(1) \setminus \{1\}, k \in \{1, 2, \ldots, n\}, \) then

\[
\frac{n}{(n-1)^2} + \sum_{k=1}^{n} \frac{x_k^2}{1-x_k} \geq \frac{2n-1}{(n-1)^2} \sum_{k=1}^{n} x_k.
\] (2.5)

**Proof.** If \( x < 1 \), the inequality \( \frac{x^2}{1-x} \geq \frac{(2n-1)x-1}{(n-1)^2} \) is equivalent with \((nx-1)^2 \geq 0\), which is a true inequality.

Using this inequality, we have that \( \sum_{k=1}^{n} \frac{x_k^2}{1-x_k} \geq \sum_{k=1}^{n} \frac{(2n-1)(x_k-1)}{(n-1)^2} \), from where (2.5) results.

**Corollary 2.3.** If \( n \in \mathbb{N} \setminus \{1\} \) and \( y_k \in D_1(0), k \in \{1, 2, \ldots, n\} \), \( y_2 + y_3 + \cdots + y_n > 0, y_3 + y_4 + \cdots + y_1 > 0, \ldots, y_1 + y_2 + \cdots + y_{n-1} > 0 \) then

\[
\sum_{cyclic} \frac{y_1^2}{y_2 + y_3 + \cdots + y_n} \geq \frac{1}{n-1} \sum_{k=1}^{n} y_k.
\] (2.6)

**Proof.** In Theorem 2.3 we take \( x_k = \frac{y_k}{y_1 + y_2 + \cdots + y_n} \), \( k \in \{1, 2, \ldots, n\} \).

**Corollary 2.4.** If \( y_1, y_2, y_3 \in D_1(0), y_1 + y_2 > 0, y_2 + y_3 > 0 \) and \( y_3 + y_1 > 0 \), then

\[
\sum_{cyclic} \frac{y_1^2}{y_2 + y_3} \geq \frac{y_1 + y_2 + y_3}{2}
\] (2.7)

(OIM 1995)

**Proof.** In Corollary 2.3 we take \( n = 3 \).

**Theorem 2.4.** If \( n \in \mathbb{N}, n \geq 3 \) and \( x_k \in D_1 \left( \frac{n^2 - 4n + 1}{2(n^2 - n - 2)} \right), k \in \{1, 2, \ldots, n\}, \) then

\[
\sum_{k=1}^{n} \frac{(x_k + 1)^2}{3x_k^2 - 2x_k + 1} \leq \frac{4n^2(n^2 - n - 2)x + n^4 - 4n^3 + n^2 + 24n}{(n^2 - 2n + 3)^2}.
\] (2.8)

**Proof.** It is verified that the inequalities

\[
\frac{4x+1}{3x^2 - 2x+1} \leq \frac{6n^2(n^2 - n - 2)x + n^4 - 4n^3 + n^2 + 24n}{(n^2 - 2n + 3)^2}
\]

and \((2(n^2 - n - 2)x - (n^2 - 4n + 1))(nx - 1)^2 \geq 0\) are equivalent for all \( x \geq \frac{n^2 - 4n + 1}{2(n^2 - n - 2)}, n \in \mathbb{N}, n \geq 3 \) and the identity \( \frac{(x + 1)^2}{3x^2 - 2x + 1} = \frac{1}{3} + \frac{2(4x+1)}{3(3x^2 - 2x + 1)} \) holds for all \( x \in \mathbb{R} \). Taking into account the relations above, we have that

\[
\sum_{k=1}^{n} \frac{(x_k + 1)^2}{3x_k^2 - 2x_k + 1} = \frac{n}{3} + \frac{2}{3} \sum_{k=1}^{n} \frac{4x_k+1}{3x_k^2 - 2x_k + 1}
\]

\[
\leq \frac{4}{3} + \frac{2}{3} \sum_{k=1}^{n} \frac{6n^2(n^2 - n - 2)x_k + n^4 - 4n^3 + n^2 + 24n}{(n^2 - 2n + 3)^2},
\]

from where (2.8) results.

**Corollary 2.5.** If \( x_1, x_2, x_3 \in D_1 \left( -\frac{1}{4} \right), \) then

\[
\sum_{k=1}^{3} \frac{(x_1 + 1)^2}{3x_1^2 - 2x_1 + 1} \leq 4(x_1 + x_2 + x_3) + 4.
\] (2.9)

**Proof.** In Theorem 2.4 we take \( n = 3 \).
Corollary 2.6. If \( y_1, y_2, y_3 \in \mathbb{R}, y_1 + y_2 + y_3 \neq 0 \) and \( \frac{y_k}{y_1 + y_2 + y_3} \in D_1 \left( -\frac{1}{4}, \frac{1}{4} \right), k \in \{1, 2, 3\}, \) then
\[
\sum_{cyclic} \frac{(2y_1 + y_2 + y_3)^2}{2y_k^2 + (y_2 + y_3)^2} \leq 8. \tag{2.10}
\]

Proof. In Corollary 2.5 we take \( x_k = \frac{y_k}{y_1 + y_2 + y_3}, k \in \{1, 2, 3\} \). \( \square \)

Remark 2.1. The (2.10) inequality was given at the USA MO in 2008, but in the conditions in which \( y_1, y_2, y_3 > 0 \).

Theorem 2.5. Let \( n \in \mathbb{N} \) and \( a \in \left( 0, \frac{1}{n} \right) \cup \left( \frac{1}{n}, \infty \right) \). If \( x_k \in S_1(a) \setminus \{a\} \), then
\[
\sum_{k=1}^{n} \frac{x_k}{a - x_k} + \frac{n}{a(n - 1)^2} \geq \frac{an^2 - 1}{(an - 1)^2} \sum_{k=1}^{n} x_k. \tag{2.11}
\]

Proof. Taking into account that the inequalities \( \frac{x}{a - x} \geq \frac{an^2 x - 1}{(an - 1)^2} \) and \( a(nx - 1)^2 \geq 0 \) are equivalent for all \( x \in S_1(a) \setminus \{a\} \). \( \square \)

Corollary 2.7. Let \( n \in \mathbb{N}, n \geq 2 \) and \( a \in \left( 0, \frac{1}{n} \right) \cup \left( \frac{1}{n}, \infty \right) \).

If \( y_1, y_2, \ldots, y_n \in \mathbb{R}, y_1 + y_2 + \cdots + y_n \neq 0 \) and \( \frac{y_k}{y_1 + y_2 + \cdots + y_n} < a, \) then \( \sum_{k=1}^{n} \frac{y_k}{a(y_1 + y_2 + \cdots + y_{k-1} + y_{k+1} + \cdots + y_n) + (a - 1)y_k} \geq \frac{n}{an - 1} \geq \frac{n}{n - 1} \). \( \tag{2.12} \)

Proof. In Theorem 2.5 we take \( x_k = \frac{y_k}{y_1 + y_2 + \cdots + y_n}, k \in \{1, 2, \ldots, n\} \). \( \square \)

Corollary 2.8. If \( n \in \mathbb{N}, n \geq 2, y_1, y_2, \ldots, y_n \in \mathbb{R}, y_1 + y_2 + \cdots + y_n \neq 0 \) and \( \frac{y_k}{y_1 + y_2 + \cdots + y_n} < 1, k \in \{1, 2, \ldots, n\} \), then
\[
\sum_{k=1}^{n} \frac{y_k}{y_1 + y_2 + \cdots + y_{k-1} + y_{k+1} + \cdots + y_n} \geq \frac{n}{n - 1}. \tag{2.13}
\]

Proof. In Corollary 2.7 we take \( a = 1 \). \( \square \)

Remark 2.2. For \( n = 3 \) we obtain Nesbitt’s inequality (see [2] or [3]),
\[
\frac{y_1}{y_2 + y_3} + \frac{y_2}{y_3 + y_1} + \frac{y_3}{y_1 + y_2} \geq \frac{3}{2}, \tag{2.14}
\]
so inequality (2.13) is a generalization of Nesbitt’s inequality.

Theorem 2.6. If \( x_k \in S_1 \left( \frac{4}{3} \right) \setminus \{1\}, k \in \{1, 2, \ldots, n\} \), then
\[
\sum_{k=1}^{n} \left( \frac{2x_k}{1 - x_k} \right)^{\frac{2}{3}} \geq 3 \sum_{k=1}^{n} x_k. \tag{2.15}
\]

Proof. If \( x \leq \frac{4}{3} \), the inequality \( \left( \frac{2x}{1 - x} \right)^{\frac{2}{3}} \geq 3x \) is equivalent with \( (3x - 1)^2(3x - 4) \leq 0 \). \( \square \)

Corollary 2.9. If \( a, b, c \in D_1(0) \setminus \{0\}, \) then
\[
\sum_{cyclic} \left( \frac{2a}{b + c} \right)^{\frac{2}{3}} \geq 3. \tag{2.16}
\]

USA MO 2002

Proof. In Theorem 2.6 we take \( x = \frac{a}{a + b + c}, y = \frac{b}{a + b + c} \) and \( z = \frac{c}{a + b + c} \). \( \square \)
**Theorem 2.7.** If \( x_k \in (D_1(2) \cap S_1(3)) \setminus \{3\}, k \in \{1, 2, \ldots, n\}, \) then

\[
5n + \sum_{k=1}^{n} \frac{x_k^2}{x_k - 3} \leq \frac{1}{2} \sum_{k=1}^{n} x_k.
\]

(2.17)

**Proof.** If \( x \in (D_1(2) \cap S_1(3)) \setminus \{3\}, \) the inequality \( \frac{x^2}{x - 3} \leq \frac{1}{2} x - 5 \) is equivalent with \( (x - 2)(x + 15) \geq 0, \) which is a true inequality.

Using the inequality above, we have that

\[
\sum_{k=1}^{n} \frac{x_k^2}{x_k - 3} \leq \sum_{k=1}^{n} \left( \frac{1}{2} x_k - 5 \right),
\]

from where (2.17) results. \( \square \)

**Theorem 2.8.** If \( x_k \in D_1(-1) \cap S_1(2), k \in \{1, 2, \ldots, n\}, \) then

\[
\sum_{\text{cyclic}} \frac{x_1^2}{3 - x_2} \leq 2n + \sum_{k=1}^{n} x_k.
\]

(2.18)

**Proof.** If \( x, y \in D_1(-1) \cap S_1(2), \) then the inequality \( \frac{x^2}{3 - y} \leq x + 2 \) is equivalent with \( (x + 1)(x - 2) + (x + 2)(y - 2) \leq 0, \)

which is a true inequality. \( \square \)

### 3. Inequalities in \( D_2 \) and \( S_2 \)

In this section, we start with geometrical images for some particular domain of \( D_2 \) type.

**Example 3.1.** The set \( D_2(2, 0) = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 2x + y \geq 0\}. \)

**Example 3.2.** The set \( D_2(1, 0) = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x + y \geq 0\}. \)

**Example 3.3.** The set \( D_2(a, 0) \cap D_2 \left( \frac{1}{a}, 0 \right), \) where \( a \neq 0. \)

a) if \( a > 0 \)
Theorem 3.9. If \( n \in \mathbb{N}, n \geq 2 \) and \((x_1, x_2), (x_2, x_3), \ldots, (x_n, x_1) \in D_2(2, 0), x_1 + x_2 \neq 0, x_2 + x_3 \neq 0, \ldots, x_n + x_1 \neq 0,\) then
\[
\sum_{cyclic} \frac{x_1^3}{(x_1 + x_2)^2} \geq \frac{1}{4} \sum_{k=1}^{n} x_k. \tag{3.1}
\]

Proof. It is verified that the inequalities \( \frac{x^3}{(x+y)^2} \geq \frac{2x-y}{4} \) and \((x-y)^2(2x+y) \geq 0\) are equivalent for all \((x, y) \in D_2(2, 0)\) and because the second inequality is true, it results that the first inequality is true. Then we have that
\[
\sum_{cyclic} \frac{x_1^3}{(x_1 + x_2)^2} \geq \sum_{cyclic} \frac{2x_1 - x_2}{4} = \frac{1}{4} \sum_{k=1}^{n} x_k,
\]
so (3.1) is obtained. \(\square\)

Remark 3.1. In Example 3.1 we see that the domain for which the inequality (3.1) holds. This set includes the set \( \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0, y \geq 0\}\). 

Corollary 3.1. If \( n \in \mathbb{N}, n \geq 2 \) and \((y_1, y_2), (y_2, y_3), \ldots, (y_n, y_1) \in D_2(2, 3), y_1 + y_2 \neq 2, y_2 + y_3 \neq 2, \ldots, y_n + y_1 \neq 2,\) then
\[
\sum_{cyclic} \frac{(y_1 - 1)^3}{(y_1 + y_2 - 2)^2} \geq \frac{1}{4} \sum_{k=1}^{n} y_k - \frac{n}{4}. \tag{3.2}
\]

Proof. In Theorem 3.9 we take \( x_k = y_k - 1, k \in \{1, 2, \ldots, n\}. \) \(\square\)

Theorem 3.10. If \( n \in \mathbb{N}, n \geq 2 \) and
\((x_1, x_2), (x_2, x_3), \ldots, (x_n, x_1) \in (S_2(3, 0) \cap D_2(1, 0))\setminus\{(0, 0)\},\) then
\[
\sum_{cyclic} \frac{x_1^2}{x_1 + x_2} \leq \frac{1}{2} \sum_{k=1}^{n} x_k^2. \tag{3.3}
\]

Proof. We start from the inequality \( \frac{x^3}{x+y} \leq \frac{5x^2 - y^2}{8} \) for all \((x, y) \in (S_2(3, 0) \cap D_2(1, 0))\setminus\{(0, 0)\}.
\]
This inequality is true because it is equivalent with the following true inequality \((x - y)^2(3x + y) \leq 0\) for all \((x, y) \in (S_2(3, 0) \cap D_2(1, 0))\setminus\{(0, 0)\}. \) \(\square\)
Remark 3.2. Let \( x^2 + axy + y^2, x, y \in \mathbb{R} \) be a polynomial. If \( a \in (-2, 2) \) then \( \Delta = a^2 - 4 < 0 \), so \( x^2 + axy + y^2 > 0 \) for all \( (x, y) \in \mathbb{R} \times \mathbb{R} \setminus \{(0, 0)\} \).

If \( a = 2 \), then \( x^2 + 2xy + y^2 = (x + y)^2 \) and this case is studied in Theorem 3.9.

In the following let \( a \in (-2, 2) \).

**Theorem 3.11.** If \( n \in \mathbb{N}, n \geq 2 \) and

\[
(x_1, x_2), (x_2, x_3), \ldots, (x_n, x_1) \in D_2(a, 0) \setminus \{(0, 0)\},
\]

then

\[
\sum_{cyclic} \frac{x_1^3}{x_1^2 + ax_1x_2 + x_2^2} \geq \frac{1}{a+2} \sum_{k=1}^{n} x_k. \tag{3.4}
\]

**Proof.** The following inequalities \( \frac{x_3^3}{x_2^2 + ax_2x_3 + x_3^2} \geq \frac{2x_3 - y}{a+2} \) and \((x - y)^2(a + y) \geq 0\) for all \((x, y) \in D_2(a, 0) \setminus \{(0, 0)\}\). Then

\[
\sum_{cyclic} \frac{x_1^3}{x_1^2 + ax_1x_2 + x_2^2} \geq \sum_{cyclic} \frac{2x_1 - x_2}{a+2}, \text{ from where the inequality (3.4) results.} \]

**Remark 3.3.** The inequalities used in the proofs from Theorem 3.10 and Theorem 3.11 appear in [1], where, by particularizations, the author M. Bencze obtains some inequalities in triangles.

**Theorem 3.12.** If \( n \in \mathbb{N}, n \geq 2, a \neq 0 \) and

\[
(x_1, x_2), (x_2, x_3), \ldots, (x_n, x_1) \in \left(D_2(a, 0) \cap D_2\left(\frac{1}{a}, 0\right)\right) \setminus \{(0, 0)\},
\]

then

\[
\sum_{cyclic} \frac{x_1^3 + x_2^3}{x_1^2 + ax_1x_2 + x_2^2} \geq \frac{2}{a+2} \sum_{k=1}^{n} x_k. \tag{3.5}
\]

**Proof.** Taking Theorem 3.11 into account, we have that

\[
\sum_{cyclic} \frac{x_1^3}{x_1^2 + ax_1x_2 + x_2^2} \geq \frac{1}{a+2} \sum_{k=1}^{n} x_k
\]

and

\[
\sum_{cyclic} \frac{x_2^3}{x_1^2 + ax_1x_2 + x_2^2} \geq \frac{1}{a+2} \sum_{k=1}^{n} x_k,
\]

from where the inequality (3.5) results. \( \square \)

**REFERENCES**

