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New category of inequalities

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ABSTRACT.
In this paper we present a new category of inequalities, which gives new refinement for some classical inequalities.

1. INTRODUCTION

In the following, we consider the sets

$$D_1(a) = \{x \in \mathbb{R} : x \geq a\}, \qquad S_1(a) = \{x \in \mathbb{R} : x \leq a\},$$
$$D_2(a,b) = \{(x,y) \in \mathbb{R}^2 : ax + y \geq b\}, \quad S_2(a,b) = \{(x,y) \in \mathbb{R}^2 : ax + y \leq b\},$$

where $a, b \in \mathbb{R}$ and let $\mathbb{N} = \{1, 2, 3, \dots\}$.
These offer light conditions for some classical inequalities.

2. INEQUALITIES IN D_1 AND S_1

Theorem 2.1. *If $n \in \mathbb{N}$, $x_k \in D_1(0)$, $k \in \{1, 2, \dots, n\}$ and $\alpha \geq 1$, then*

$$\sum_{k=1}^{n} \frac{x_k}{1 + (\alpha - 1)x_k} \leq \frac{n^2}{(n + \alpha - 1)^2} \sum_{k=1}^{n} x_k + \frac{n(\alpha - 1)}{(n + \alpha - 1)^2}. \tag{2.1}$$

Proof. If $x > 0$ and $\alpha \geq 1$, the inequality $\frac{x}{1 + (\alpha - 1)x} \leq \frac{n^2 x + \alpha - 1}{(n + \alpha - 1)^2}$ holds, because it is equivalent with $(\alpha - 1)(nx - 1)^2 \geq 0$. For $x \in \{x_1, x_2, \dots, x_n\}$ and summing the inequalities obtained, we have that

$$\sum_{k=1}^{n} \frac{x_k}{1 + (\alpha - 1)x_k} \leq \sum_{k=1}^{n} \frac{n^2 x_k + \alpha - 1}{(n + \alpha - 1)^2},$$

from where (2.1) results. □

Corollary 2.1. *If $n \in \mathbb{N}$, $\alpha \geq 1$, $y_k \in D_1(0)$, $k \in \{1, 2, \dots, n\}$ and $y_1 + y_2 + \dots + y_n \neq 0$, then*

$$\sum_{cyclic} \frac{y_1}{\alpha y_1 + y_2 + \dots + y_n} \leq \frac{n}{n + \alpha - 1}. \tag{2.2}$$

Proof. In Theorem 2.1 we take $x_k = \frac{y_k}{y_1 + y_2 + \dots + y_n}$, $k \in \{1, 2, \dots, n\}$. □

Theorem 2.2. *If $n \in \mathbb{N} \setminus \{2\}$ and $x_k \in S_1\left(\frac{1}{2}\right) \setminus \left\{\frac{1}{2}\right\}$, $k \in \{1, 2, \dots, n\}$, then*

$$\frac{2n}{(n-2)^2} + \sum_{k=1}^{n} \frac{x_k}{1 - 2x_k} \geq \left(\frac{n}{n-2}\right)^2 \sum_{k=1}^{n} x_k. \tag{2.3}$$

Proof. If $x < \frac{1}{2}$ and $n \in \mathbb{N} \setminus \{2\}$, the inequality $\frac{x}{1 - 2x} \geq \frac{n^2 x - 2}{(n-2)^2}$ is true because it is equivalent with $(nx - 1)^2 \geq 0$. Taking the inequality above account, we have that $\sum_{k=1}^{n} \frac{x_k}{1 - 2x_k} \geq \sum_{k=1}^{n} \frac{n^2 x_k - 2}{(n-2)^2}$, from where (2.3) results. □

Corollary 2.2. *Let $n \in \mathbb{N}$, $n \geq 3$. In all convex polygons with sides $y_1, y_2, \dots, y_n$, the inequality*

$$\sum_{cyclic} \frac{y_1}{y_2 + y_3 + \dots + y_n - y_1} \geq \frac{n}{n-2} \tag{2.4}$$

holds.

Proof. In Theorem 2.2 we take $x_k = \frac{y_k}{y_1 + y_2 + \dots + y_n}$, $k \in \{1, 2, \dots, n\}$ and it is verified immediately that $x_k < \frac{1}{2}$, $k \in \{1, 2, \dots, n\}$ because $y_1, y_2, \dots, y_n$ are the sides of a convex polygon. □

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Theorem 2.3. *If $n \in \mathbb{N}\backslash\{1\}$ and $x_k \in S_1(1)\backslash\{1\}$, $k \in \{1, 2, \dots, n\}$, then*

$$\frac{n}{(n-1)^2} + \sum_{k=1}^{n} \frac{x_k^2}{1-x_k} \geq \frac{2n-1}{(n-1)^2} \sum_{k=1}^{n} x_k. \tag{2.5}$$

Proof. If $x < 1$, the inequality $\dfrac{x^2}{1-x} \geq \dfrac{(2n-1)x-1}{(n-1)^2}$ is equivalent with $(nx-1)^2 \geq 0$, which is a true inequality. Using this inequality, we have that $\displaystyle\sum_{k=1}^{n} \frac{x_k^2}{1-x_k} \geq \sum_{k=1}^{n} \frac{(2n-1)(x_k-1)}{(n-1)^2}$, from where (2.5) results. □

Corollary 2.3. *If $n \in \mathbb{N}\backslash\{1\}$ and $y_k \in D_1(0)$, $k \in \{1, 2, \dots, n\}$, $y_2 + y_3 + \cdots + y_n > 0$, $y_3 + y_4 + \cdots + y_1 > 0, \dots, y_1 + y_2 + \cdots + y_{n-1} > 0$ then*

$$\sum_{cyclic} \frac{y_1^2}{y_2 + y_3 + \cdots + y_n} \geq \frac{1}{n-1} \sum_{k=1}^{n} y_k. \tag{2.6}$$

Proof. In Theorem 2.3 we take $x_k = \dfrac{y_k}{y_1 + y_2 + \cdots + y_n}$, $k \in \{1, 2, \dots, n\}$. □

Corollary 2.4. *If $y_1, y_2, y_3 \in D_1(0)$, $y_1 + y_2 > 0$, $y_2 + y_3 > 0$ and $y_3 + y_1 > 0$, then*

$$\sum_{cyclic} \frac{y_1^2}{y_2 + y_3} \geq \frac{y_1 + y_2 + y_3}{2} \tag{2.7}$$

(OIM 1995)

Proof. In Corollary 2.3 we take $n = 3$. □

Theorem 2.4. *If $n \in \mathbb{N}$, $n \geq 3$ and $x_k \in D_1\left(\dfrac{n^2-4n+1}{2(n^2-n-2)}\right)$, $k \in \{1, 2, \dots, n\}$, then*

$$\sum_{k=1}^{n} \frac{(x_k+1)^2}{3x_k^2 - 2x_k + 1} \leq \frac{4n^2(n^2-n-2)}{(n^2-2n+3)^2} \sum_{k=1}^{n} x_k + \frac{n(n^4 - 4n^3 + 4n^2 + 12n + 3)}{(n^2-2n+3)^2}. \tag{2.8}$$

Proof. It is verified that the inequalities

$$\frac{4x+1}{3x^2-2x+1} \leq \frac{6n^2(n^2-n-2)x + n^4 - 4n^3 + n^2 + 24n}{(n^2-2n+3)^2}$$

and $(2(n^2-n-2)x - (n^2-4n+1))(nx-1)^2 \geq 0$ are equivalent for all $x \geq \dfrac{n^2-4n+1}{2(n^2-n-2)}$, $n \in \mathbb{N}$, $n \geq 3$ and the identity $\dfrac{(x+1)^2}{3x^2-2x+1} = \dfrac{1}{3} + \dfrac{2(4x+1)}{3(3x^2-2x+1)}$ holds for all $x \in \mathbb{R}$. Taking into account the relations above, we have that

$$\begin{aligned}\sum_{k=1}^{n} \frac{(x_k+1)^2}{3x_k^2-2x_k+1} &= \frac{n}{3} + \frac{2}{3}\sum_{k=1}^{n} \frac{4x_k+1}{3x_k^2-2x_k+1} \\ &\leq \frac{4}{3} + \frac{2}{3}\sum_{k=1}^{n} \frac{6n^2(n^2-n-2)x_k + n^4 - 4n^3 + n^2 + 24n}{(n^2-2n+3)^2},\end{aligned}$$

from where (2.8) results. □

Corollary 2.5. *If $x_1, x_2, x_3 \in D_1\left(-\dfrac{1}{4}\right)$, then*

$$\sum_{k=1}^{3} \frac{(x_1+1)^2}{3x_1^2 - 2x_1 + 1} \leq 4(x_1 + x_2 + x_3) + 4. \tag{2.9}$$

Proof. In Theorem 2.4 we take $n = 3$. □

Corollary 2.6. *If* $y_1, y_2, y_3 \in \mathbb{R}$, $y_1 + y_2 + y_3 \neq 0$ *and* $\dfrac{y_k}{y_1 + y_2 + y_3} \in D_1\left(-\dfrac{1}{4}\right)$, $k \in \{1, 2, 3\}$, *then*

$$\sum_{cyclic} \frac{(2y_1 + y_2 + y_3)^2}{2y_1^2 + (y_2 + y_3)^2} \leq 8. \tag{2.10}$$

Proof. In Corollary 2.5 we take $x_k = \dfrac{y_k}{y_1 + y_2 + y_3}$, $k \in \{1, 2, 3\}$. □

Remark 2.1. The (2.10) inequality was given at the USA MO in 2008, but in the conditions in which $y_1, y_2, y_3 > 0$.

Theorem 2.5. *Let* $n \in \mathbb{N}$ *and* $a \in \left(0, \dfrac{1}{n}\right) \cup \left(\dfrac{1}{n}, \infty\right)$. *If* $x_k \in S_1(a))\backslash\{a\}$, *then*

$$\sum_{k=1}^{n} \frac{x_k}{a - x_k} + \frac{n}{(an - 1)^2} \geq \frac{an^2}{(an - 1)^2} \sum_{k=1}^{n} x_k. \tag{2.11}$$

Proof. Taking into account that the inequalities $\dfrac{x}{a - x} \geq \dfrac{an^2 x - 1}{(an - 1)^2}$ and $a(nx - 1)^2 \geq 0$ are equivalent for all $x \in S_1(a))\backslash\{a\}$. □

Corollary 2.7. *Let* $n \in \mathbb{N}$, $n \geq 2$ *and* $a \in \left(0, \dfrac{1}{n}\right) \cup \left(\dfrac{1}{n}, \infty\right)$.

If $y_1, y_2, \ldots, y_n \in \mathbb{R}$, $y_1 + y_2 + \cdots + y_n \neq 0$ *and* $\dfrac{y_k}{y_1 + y_2 + \cdots + y_n} < a$, $k \in \{1, 2, \ldots, n\}$, *then*

$$\sum_{k=1}^{n} \frac{y_k}{a(y_1 + y_2 + \cdots + y_{k-1} + y_{k+1} + \cdots + y_n) + (a - 1)y_k} \geq \frac{n}{an - 1}. \tag{2.12}$$

Proof. In Theorem 2.5 we take $x_k = \dfrac{y_k}{y_1 + y_2 + \cdots + y_n}$, $k \in \{1, 2, \ldots, n\}$. □

Corollary 2.8. *If* $n \in \mathbb{N}$, $n \geq 2$, $y_1, y_2, \ldots, y_n \in \mathbb{R}$, $y_1 + y_2 + \cdots + y_n \neq 0$ *and* $\dfrac{y_k}{y_1 + y_2 + \cdots + y_n} < 1$, $k \in \{1, 2, \ldots, n\}$, *then*

$$\sum_{k=1}^{n} \frac{y_k}{y_1 + y_2 + \cdots + y_{k-1} + y_{k+1} + \cdots + y_n} \geq \frac{n}{n - 1}. \tag{2.13}$$

Proof. In Corollary 2.7 we take $a = 1$. □

Remark 2.2. For $n = 3$ we obtain Nesbitt's inequality (see [2] or [3]),

$$\frac{y_1}{y_2 + y_3} + \frac{y_2}{y_3 + y_1} + \frac{y_3}{y_1 + y_2} \geq \frac{3}{2}, \tag{2.14}$$

so inequality (2.13) is a generalization of Nesbitt's inequality.

Theorem 2.6. *If* $x_k \in S_1\left(\dfrac{4}{3}\right)\backslash\{1\}$, $k \in \{1, 2, \ldots, n\}$, *then*

$$\sum_{k=1}^{n} \left(\frac{2x_k}{1 - x_k}\right)^{\frac{2}{3}} \geq 3 \sum_{k=1}^{n} x_k. \tag{2.15}$$

Proof. If $x \leq \dfrac{4}{3}$, the inequality $\left(\dfrac{2x}{1 - x}\right)^{\frac{2}{3}} \geq 3x$ is equivalent with $(3x - 1)^2(3x - 4) \leq 0$. □

Corollary 2.9. *If* $a, b, c \in D_1(0)\backslash\{0\}$, *then*

$$\sum_{cyclic} \left(\frac{2a}{b + c}\right)^{\frac{2}{3}} \geq 3 \tag{2.16}$$

(USA MO 2002)

Proof. In Theorem 2.6 we take $x = \dfrac{a}{a + b + c}$, $y = \dfrac{b}{a + b + c}$ and $z = \dfrac{c}{a + b + c}$. □

Theorem 2.7. *If $x_k \in (D_1(2) \cap S_1(3)) \setminus \{3\}$, $k \in \{1, 2, \dots, n\}$, then*

$$5n + \sum_{k=1}^{n} \frac{x_k^2}{x_k - 3} \leq \frac{1}{2} \sum_{k=1}^{n} x_k. \tag{2.17}$$

Proof. If $x \in (D_1(2) \cap S_1(3)) \setminus \{3\}$, the inequality $\frac{x^2}{x-3} \leq \frac{1}{2}x - 5$ is equivalent with $(x-2)(x+15) \geq 0$, which is a true inequality.

Using the inequality above, we have that $\sum_{k=1}^{n} \frac{x_k^2}{x_k - 3} \leq \sum_{k=1}^{n} \left(\frac{1}{2}x_k - 5 \right)$, from where (2.17) results. □

Theorem 2.8. *If $x_k \in D_1(-1) \cap S_1(2)$, $k \in \{1, 2, \dots, n\}$, then*

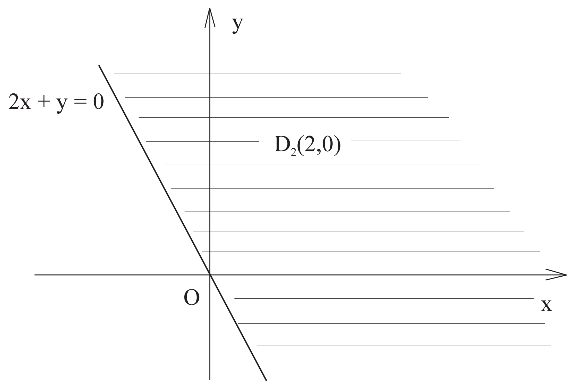
$$\sum_{cyclic} \frac{x_1^2}{3 - x_2} \leq 2n + \sum_{k=1}^{n} x_k. \tag{2.18}$$

Proof. If $x, y \in D_1(-1) \cap S_1(2)$, then the inequality $\frac{x^2}{3-y} \leq x + 2$ is equivalent with $(x+1)(x-2) + (x+2)(y-2) \leq 0$, which is a true inequality. □

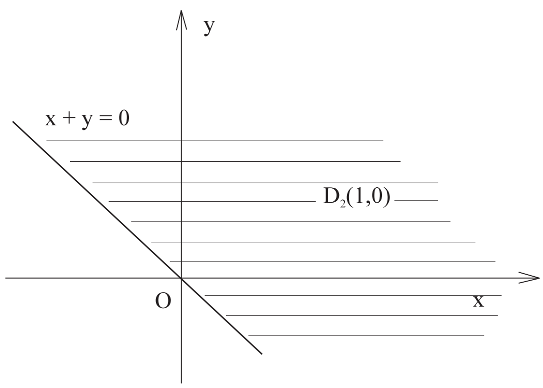
3. Inequalities in D_2 and S_2

In this section, we start with geometrical images for some particular domain of D_2 type.

Example 3.1. The set $D_2(2,0) = \{(x,y) \in \mathbb{R} \times \mathbb{R} : 2x + y \geq 0\}$.

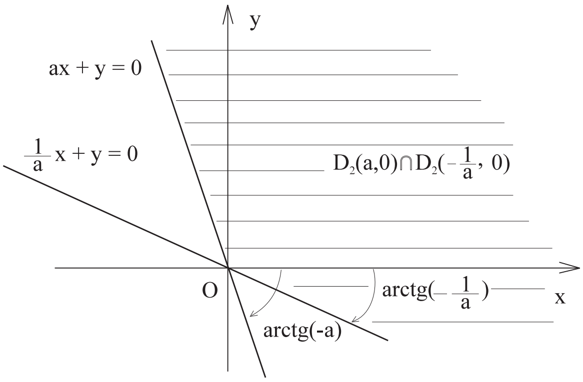


Example 3.2. The set $D_2(1,0) = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x + y \geq 0\}$.

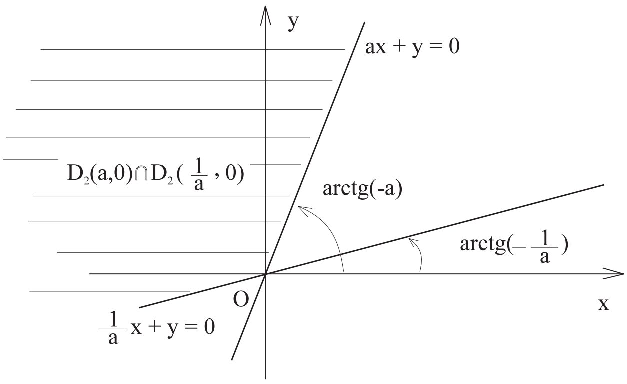


Example 3.3. The set $D_2(a,0) \cap D_2\left(\frac{1}{a}, 0\right)$, where $a \neq 0$.

a) if $a > 0$



b) if $a < 0$



Theorem 3.9. *If $n \in \mathbb{N}$, $n \geq 2$ and $(x_1, x_2), (x_2, x_3), \ldots, (x_n, x_1) \in D_2(2, 0)$, $x_1 + x_2 \neq 0$, $x_2 + x_3 \neq 0, \ldots, x_n + x_1 \neq 0$, then*

$$\sum_{cyclic} \frac{x_1^3}{(x_1 + x_2)^2} \geq \frac{1}{4} \sum_{k=1}^{n} x_k. \tag{3.1}$$

Proof. It is verified that the inequalities $\frac{x^3}{(x+y)^2} \geq \frac{2x-y}{4}$ and $(x-y)^2(2x+y) \geq 0$ are equivalent for all $(x, y) \in D_2(2, 0)$ and because the second inequality is true, it results that the first inequality is true. Then we have that

$$\sum_{cyclic} \frac{x_1^3}{(x_1 + x_2)^2} \geq \sum_{cyclic} \frac{2x_1 - x_2}{4} = \frac{1}{4} \sum_{k=1}^{n} x_k,$$

so (3.1) is obtained. □

Remark 3.1. In Example 3.1 we see that the domain for which the inequality (3.1) holds. This set includes the set $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0, y \geq 0\} \backslash \{(0, 0)\}$.

Corollary 3.1. *If $n \in \mathbb{N}$, $n \geq 2$ and $(y_1, y_2), (y_2, y_3), \ldots, (y_n, y_1) \in D_2(2, 3)$, $y_1 + y_2 \neq 2$, $y_2 + y_3 \neq 2, \ldots, y_n + y_1 \neq 2$, then*

$$\sum_{cyclic} \frac{(y_1 - 1)^3}{(y_1 + y_2 - 2)^2} \geq \frac{1}{4} \sum_{k=1}^{n} y_k - \frac{n}{4}. \tag{3.2}$$

Proof. In Theorem 3.9 we take $x_k = y_k - 1$, $k \in \{1, 2, \ldots, n\}$. □

Theorem 3.10. *If $n \in \mathbb{N}$, $n \geq 2$ and*

$$(x_1, x_2), (x_2, x_3), \ldots, (x_n, x_1) \in (S_2(3, 0) \cap D_2(1, 0)) \backslash \{(0, 0)\},$$

then

$$\sum_{cyclic} \frac{x_1^3}{x_1 + x_2} \leq \frac{1}{2} \sum_{k=1}^{n} x_k^2. \tag{3.3}$$

Proof. We start from the inequality $\frac{x^3}{x + y} \leq \frac{5x^2 - y^2}{8}$ for all

$$(x, y) \in (S_2(3, 0) \cap D_2(1, 0)) \backslash \{(0, 0)\}.$$

This inequality is true because it is equivalent with the following true inequality $(x - y)^2(3x + y) \leq 0$ for all $(x, y) \in (S_2(3, 0) \cap D_2(1, 0) \backslash \{(0, 0)\}$. □

Remark 3.2. Let $x^2 + axy + y^2$, $x, y \in \mathbb{R}$ be a polynomial. If $a \in (-2, 2)$ then $\Delta = a^2 - 4 < 0$, so $x^2 + axy + y^2 > 0$ for all $(x, y) \in \mathbb{R} \times \mathbb{R} \backslash \{(0, 0)\}$.
If $a = 2$, then $x^2 + 2xy + y^2 = (x + y)^2$ and this case is studied in Theorem 3.9.

In the following let $a \in (-2, 2)$.

Theorem 3.11. *If $n \in \mathbb{N}$, $n \geq 2$ and*

$$(x_1, x_2), (x_2, x_3), \dots, (x_n, x_1) \in D_2(a, 0) \backslash \{(0, 0)\},$$

then

$$\sum_{cyclic} \frac{x_1^3}{x_1^2 + ax_1x_2 + x_2^2} \geq \frac{1}{a+2} \sum_{k=1}^{n} x_k. \tag{3.4}$$

Proof. The following inequalities $\frac{x^3}{x^2+axy+y^2} \geq \frac{2x-y}{a+2}$ and $(x-y)^2(ax+y) \geq 0$ for all $(x, y) \in D_2(a, 0) \backslash \{(0, 0)\}$. Then $\sum_{cyclic} \frac{x_1^3}{x_1^2 + ax_1x_2 + x_2^2} \geq \sum_{cyclic} \frac{2x_1 - x_2}{a+2}$, from where the inequality (3.4) results. □

Remark 3.3. The inequalities used in the proofs from Theorem 3.10 and Theorem 3.11 appear in [1], where, by particularizations, the author M. Bencze obtains some inequalities in triangles.

Theorem 3.12. *If $n \in \mathbb{N}$, $n \geq 2$, $a \neq 0$ and*

$$(x_1, x_2), (x_2, x_3), \dots, (x_n, x_1) \in \left(D_2(a, 0) \cap D_2\left(\frac{1}{a}, 0\right)\right) \backslash \{(0, 0)\},$$

then

$$\sum_{cyclic} \frac{x_1^3 + x_2^3}{x_1^2 + ax_1x_2 + x_2^2} \geq \frac{2}{a+2} \sum_{k=1}^{n} x_k. \tag{3.5}$$

Proof. Taking Theorem 3.11 into account, we have that

$$\sum_{cyclic} \frac{x_1^3}{x_1^2 + ax_1x_2 + x_2^2} \geq \frac{1}{a+2} \sum_{k=1}^{n} x_k$$

and

$$\sum_{cyclic} \frac{x_2^3}{x_1^2 + ax_1x_2 + x_2^2} \geq \frac{1}{a+2} \sum_{k=1}^{n} x_k,$$

from where the inequality (3.5) results. □

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