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Right simple injective FGF-ring

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ABSTRACT.
A ring R is called right FGF-ring if every finitely generated right R-module embeds in a free (projective) . A ring is called right simple-injective if R_R is simple R-injective, that is, if I is a right ideal of R and $\gamma : I \to R$ is an R-morphism with simple image, then $\gamma(x) = c.x$, is left multiplication by an element $c \in R$. There is a conjecture due to Carl Faith which asserts that every right FGF-ring is a Quasi-Frobenius ring (QF). In this paper we establish the conjecture in case that the ring is a simple injective ring by showing that the right simple-injective FGF ring is a right self- injective.

1. INTRODUCTION

A ring R is called a right FGF-ring if every finitely generated right R-module embeds in a free (projective) R-module. A ring is called right simple injective if R_R is simple R-injective equivalently if I is a right ideal of R and $\gamma : I \to R$ is an R-morphism with simple image, then $\gamma(x) = c.x$ is left multiplication by an element $c \in R$. A ring R is called quasi-Frobenius if R is left artinian and R is left or right self-injective (30.7. [1]).
There is a conjecture due to Carl Faith which asserts that every right FGF-ring is Quasi-Frobenius ring (QF). In this paper we establish the conjecture in case that the ring is simple injective ring.

There is a theorem of Faith and Walker that if every right R-module embeds in a free, then R is quasi-frobenius ring, the FGF-conjecture asserts that the hypothesis can be weekend to finitely generated right R-module. Right FGF-rings have been studied by Faith, Gomes Pardo and Guil Asensio. The conjunction was established in the following cases.

(1)The ring is left Kasch
Every left Kasch, right FGF-ring is quasi-frobenius (Kato [7]-[11]);(where a ring R is called left Kasch if every simple left R-module is isomorphic to a minimal left ideal of R).

(2)The ring is right perfect
Every right perfect, right FGF-ring is quasi-frobenius (Rutter [10]); (where a ring R is called right perfect if every flat right R-module is projective, equivalently if every right R-module has a projective cover).

(3)Every right selfinjective, right FGF-ring is quasi-frobenius (Bjork [2] also Osofsky [9]); (where a ring R is called right selfinjective if every right R-homomorphism from a right ideal of R can be extended to all of R).

(4) R is right continuous
Every right continuous, right FGF-ring is quasi-frobenius (Gomez Pardo and Guil Asensio [3]); (where a ring R is called right continuous (right CS ring) if every right ideal of R is essential in a direct summand of R.

The following two lemmas will be needed in our investigation

Lemma 1.1. ([6], 12.2.3). *Let R be an arbitrary ring.
If*

$$0 \to A \xrightarrow{f} M \xrightarrow{g} W \to 0$$

is a split exact sequence of right R-modules then we have M is reflexive if and only if A and W are reflexive.

Lemma 1.2. ([6], 12.2.4). *Let R be arbitrary ring. Then every finitely generated projective R-module is reflexive.*

Lemma 1.3. ([9], 2.7). *Every finitely cogenerated torsionless right R-module embeds in a free module R^n of finite rank.*

In this paper we prove that every right simple injective, right FGF-ring is QF-ring, here R is a ring with identity, and every module is a unitary right R-module.

2. MAIN RESULTS

Theorem 2.1. *If R is right simple injective FGF ring then every finitely generated right R-module is reflexive (i.e. R is a ring with perfect duality)*

Proof. **Step (1)** R is FGF ring so for each finitely generated R-module M there is a monomorphism α and a free module F such that the following sequence

$$0 \longrightarrow M \longrightarrow F$$

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is exact. Consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow[mon]{\alpha} & F \\ \varphi_M \downarrow & & \downarrow \varphi_F \\ M^{**} & \longrightarrow & F^{**} \end{array}$$

Where $M^{**} = \mathrm{Hom}_R\ ({}_R\mathrm{Hom}_R\ (A_R, R_R),_R R)$ is the bidual of A and F^{**} is the bidual of F.

Since F is a free module so φ_F is monomorphism and hence φ_M monomorphism (i.e. M is torsionless).

Step (2) since R is simple injective ring then for any ideal $I < R, cI$ is simple; for some $c \in R$. So also a_iR is simple for some $a_i \in R$.

An epimorphism $\alpha_i : R \to a_iR$ will induce the epimorphism $\alpha : \oplus_n R \to \oplus_n a_iR$. Since a_iR is simple so $\oplus_n a_iR$ is semisimple and

$$\ker\ \alpha = \{(r) \in \oplus_n R; (a_ir) = 0\}.$$

Step (3) Let M be a finitely generated module so there is an epimorphism β and a free module F such that $\beta : F \to M$. Since each finitely generated module is a factor of a free module with finite rank so

$$\beta : F \cong \oplus_n R \to M \cong \oplus_n R/k; k = ker\beta,$$

is an epimorphism given.

By

$$\beta(x_i) = \sum m_i(x_i).$$

On the other hand $m_i = (r_i) + k$. So $\beta(x_i) = \sum[(r_i) + k](x_i)$ and

$$\begin{aligned} ker\beta &= \{(x_i) \in \oplus_n R; \sum m_i(x_i) = 0\} \\ &= \{(x_i) \in \oplus_n R; (r_i)(x_i) \in k\}. \end{aligned}$$

If $(y_i) \in \ker\ \alpha$ where α is as in step (2) then

$(a_iy_i) = 0$ for some $a_i \in R$.

$$0 = \beta(0) = \beta(a_iy_i) = \sum[(r_i) + k](a_iy_i) = \sum[(r_ia_i) + k](y_i)$$

i.e. $(y_i) \in \ker\ \beta$, hence $\ker\ \alpha < \ker\ \beta$ and β induces an epimorphism

$$\bar{\beta} : \oplus_n a_iR \cong \oplus_n R/\mathrm{ker}\alpha \to M$$

hence M is epimorphic image of a semisimple module by which M is semisimple and finitely generated and hence it is finitely cogenerated and torsionless by step (1) so it can be embedded in a free module with finite rank by lemma(1.3).

Note that if $M = R_R$ then R_R and hence R need not be semisimple in general (the epimorphic image of a semisimple ring need not be semisimple) unless R_R be a module over the ring $\oplus_n R/\mathrm{ker}\alpha$ or $\bar{\beta}$ is unitary morphism

Step(4). The sequence $0 \longrightarrow M \xrightarrow[mon]{\alpha} E$ is a split monomorphism (as $\alpha(M)$ is a direct summand of E) and $\alpha(M) \cong M$ where E is a free module with finite rank which is then reflexive

Consider the split exact sequence

$$0 \longrightarrow \alpha(M) \longrightarrow E \longrightarrow E/\alpha(M) \longrightarrow 0$$

Since E is reflexive then by lemma (1.1) $\alpha(M)$ and hence $M \cong \alpha(M)$ is also reflexive, also R_R is reflexive so every finitely generated R-module is reflexive by which R is a ring with perfect duality. □

Lemma 2.1. ([6], 12.1.1) *The following are equivalent for a ring R.*
(1) Every finitely generated R-module is reflexive
(2) ${}_RR$ is a cogenerator and R_R is injective.

Since every finitely generated module is reflexive then R_R is injective by lemma (2.2), this proves the following proposition

Proposition 2.1. *Every simple injective FGF ring is right self injective.*

Since every right self injective FGF- ring is quasi-frobenius [2], [9], we obtain the following result:

Theorem 2.2. *Every simple injective FGF ring is a QF ring.*

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