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Monotone semilinear equations in Hilbert spaces and applications

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ABSTRACT.
Consider a abstract semilinear equation of the form $Au+F(u)=0$, where A is a maximal monotone map acting into a real Hilbert space H, and F is a Lipschitz strongly monotone map on H. Such equations were studied by H. Amann (1982), T. Bartsch (1988), C. Mortici and S. Sburlan (2005, 2006), D. Teodorescu (2005). By standard arguments we can reformulate the problem as a fixed point equation and prove easier some existence results. Based on these abstract results some applications to partial differential equations are also appended. The method can be adapted for teaching PDE in Technical Universities.

1. SEMILINEAR ABSTRACT EQUATIONS IN HILBERT SPACES

Consider a semilinear equation of the form

$$Au + F(u) = 0 \tag{1.1}$$

where A is a maximal monotone map acting into a real Hilbert space H and F is a Lipschitz strongly monotone map on H. Such problems were studied by Amann [1] and Bartsch [2] in the case when A is a linear self-adjoint map with the spectrum $\rho(A)$ and F is a Gâteaux differentiable map such that there exist real numbers $\nu < \mu$ such that $[\nu, \mu] \subset \rho(A)$ and

$$\nu \le \frac{< F(u) - F(v), u - v >}{|u-v|^2} \le \mu \,,\ \forall\, u, v \in H, u \neq v. \tag{1.2}$$

We have the following

Theorem 1.1. *Assume that $A : D(A) \subset H \to H$ is maximal monotone and there exist $m, M > 0$ such that*
(i) $< F(u) - F(v), u - v > \ge m \cdot |u-v|^2 \quad , \forall\, u, v \in H;$
(ii) $|F(u) - F(v)| \le M \cdot |u - v| \quad , \forall\, u, v \in H.$
Then equation (1.1) has a unique solution.

Proof. We will show that there exists $\lambda > 0$ such that $S_\lambda : H \to H, S_\lambda(u) := u - \lambda F(u)$ is a contraction. Indeed,

$$|S_\lambda(u) - S_\lambda(v)|^2 = |u-v|^2 - 2\lambda \ < F(u) - F(v), u - v > + \lambda^2 |F(u) - F(v)|^2 \le$$
$$\le (1 - 2\lambda m + \lambda^2 M)\, |u - v|^2 \,, \tag{1.3}$$

thus

$$|S_\lambda(u) - S_\lambda(v)| \le c \cdot |u - v| \tag{1.4}$$

with $c := \sqrt{1 - 2\lambda m + \lambda^2 M} < 1$, if $\lambda \in (0, \frac{2m}{M})$.

Now equation (1.1) can be written as

$$(I + \lambda A)u - (u - \lambda F(u)) = 0, \tag{1.5}$$

or

$$(I + \lambda A)u = S_\lambda(u), \tag{1.6}$$

where $\lambda > 0$ is taken as above. Using the fact that $(I + \lambda A)$ is invertible and $\left|(I + \lambda A)^{-1}\right| \le 1$ for each $\lambda > 0$ (because A is maximal monotone, (e.g. [9], p.123), equation (1.6) is equivalent with

$$u = (I + \lambda A)^{-1} S_\lambda(u). \tag{1.7}$$

We have

$$\left|(I + \lambda A)^{-1} S_\lambda(u) - (I + \lambda A)^{-1} S_\lambda(v)\right| =$$
$$= \left|(I + \lambda A)^{-1}(S_\lambda(u) - S_\lambda(v))\right| \le$$
$$\le \left|(I + \lambda A)^{-1}\right| \cdot |S_\lambda(u) - S_\lambda(v)| \le c \cdot |u - v| \ ,\ u, v \in H.$$

Therefore, $u \mapsto (I + \lambda A)^{-1} S_\lambda(u)$ is a contraction having an unique fixed point, thus (1.7) and consequently (1.1) has an unique solution. □

A similar result can be proved in the following:

Theorem 1.2. *Suppose that F satisfy (i)+(ii) and $A : D(A) \subset H \to H$ is bounded, compact and monotone. Then equation (1.1) has a unique solution.*

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Proof. Equation (1.1) can be equivalently written as

$$(\lambda I + A)u = T_\lambda(u), \tag{1.8}$$

where $T_\lambda(u) := \lambda u - F(u),\ \lambda > 0$. We have

$$|T_\lambda(u) - T_\lambda(v)|^2 = \lambda^2|u-v|^2 - 2\lambda < F(u) - F(v), u - v > +|F(u) - F(v)|^2 \leq$$
$$\leq (\lambda^2 - 2\lambda m + M^2)|u-v|^2,$$

therefore

$$|T_\lambda(u) - T_\lambda(v)| \leq \sqrt{\lambda^2 - 2\lambda m + M^2} \cdot |u - v|. \tag{1.9}$$

Let us choose $\lambda > \max\{\|A\|, \dfrac{M^2}{2m}\}$.

In particular, $\lambda > \|A\|$ implies that $\lambda I + A$ is invertible because

$$\sigma(A) \subset [-\|A\|, \|A\|].$$

Moreover,

$$|(\lambda I + A)u|^2 = \lambda^2|u|^2 + 2\lambda(Au, u) + |Au|^2 \geq \lambda^2|u|^2, \tag{1.10}$$

(because A is monotone), or

$$|(\lambda I + A)u| \geq \lambda|u|,$$

hence $|(\lambda I + A)^{-1}| \leq \frac{1}{\lambda}$. Equation (1.8) is equivalent with

$$u = (\lambda I + A)^{-1}T_\lambda(u). \tag{1.11}$$

We have

$$\left|(\lambda I + A)^{-1}T_\lambda(u) - (\lambda I + A)^{-1}T_\lambda(v)\right| = \left|(\lambda I + A)^{-1}(T_\lambda(u) - T_\lambda(v))\right| \leq$$
$$\leq \left|(\lambda I + A)^{-1}\right| \cdot |T_\lambda(u) - T_\lambda(v)| \leq \frac{1}{\lambda}\sqrt{\lambda^2 - 2\lambda m + M^2} \cdot |u - v|\,.$$

Because $\lambda > \dfrac{M^2}{2m}$, it results that

$$\gamma := \frac{1}{\lambda}\sqrt{\lambda^2 - 2\lambda m + M^2} < 1,$$

therefore $u \longmapsto (\lambda I + A)^{-1}T_\lambda(u)$ is a contraction. Now equation (1.11) and consequently (1.1) has an unique solution. □

The operator A may be a nonlinear one, but when it is a linear operator we can obtain an existence result with nice applications (see [13]):

Theorem 1.3. *Let $A : D(A) \subseteq H \to H$ be a linear map and $F : H \to H$ a nonlinear one with $Rg(F) \subseteq Rg(A)$. Suppose that there exist two positive constants $c > M$ such that:*

i) A is strongly positive of constant $c > 0$, i.e.,

$$\langle Ax, x\rangle \geq c\,\|x\|, \forall\, x \in D(A);$$

ii) F is Lipschitz continuous of constant $M > 0$, i.e.,

$$|F(x) - F(y)| \leq M\,|x - y|, \forall\, x, y \in H.$$

Then the unhomogeneous equation

$$Au + F(u) = f \tag{1.1'}$$

has a unique solution for each $f \in Rg(A)$.

Proof. Indeed, $i)$ implies that A^{-1} exists as a linear continuous map from $H_1 := Rg(A)$ into H and

$$\left\|A^{-1}\right\|_{L(H_1,H)} \leq \frac{1}{c}.$$

Then, equation (1.1′) can be equivalently rewritten as

$$V(u) := (I + A^{-1}F)u = A^{-1}f =: g. \tag{1.1''}$$

It is easily seen that V is Lipschitz continuous, i.e.,

$$|V(x) - V(y)| \leq \alpha\,|x - y|\,,\ \alpha := 1 + \frac{M}{c},\ \forall\, x, y \in H$$

and strongly monotone

$$\langle V(x) - V(y), x - y\rangle \geq \beta\,|x - y|^2\,,\ \beta := 1 - \frac{M}{c},\ \forall\, x, y \in H.$$

Thus $S_\lambda u := u - \lambda(V(u) - g) = (I - \lambda V)u + \lambda g$ is a contraction

$$|S_\lambda x - S_\lambda y| \leq \left(1 - 2\lambda\beta + \alpha^2\lambda^2\right)|x - y|^2\,,$$

for $\lambda \in \left(0, \frac{2\beta}{\alpha^2}\right)$ and we can apply the contraction mapping theorem. □

For the unique solution u^* of equation (1.1′) we have the following estimation

$$|u^*| = \left|A^{-1}(f - F(u^*))\right| \leq \frac{1}{c}|f - F(u^*)|$$

where $|F(u^*)| \leq |F(u^*) - F(0)| + |F(0)| \leq M|u^*| + |F(0)|$. Consequently we have that

$$|u^*| \leq \frac{1}{c-M}(|f| + |F(0)|).$$

This inequality allows to prove the continuous dependence of the solution on the second term in (1.1′), that is,

$$|u_1^* - u_2^*| \leq \frac{1}{c-M}|f_1 - f_2|, \ \forall\, f_1, f_2 \in Rg(A),$$

where u_i^* denote the unique solutions of the equations

$$Au + F(u) = f_i, \ i = 1, 2,$$

respectively. Therefore, under the hypothesis of Theorem 1.3, the problem (1.1″) is well posed.

2. Applications to differential equations

(a1) Semilinear Elliptic Boundary Problems ([8])
Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $a_{ij} \in C^1(\overline{\Omega}),\ 1 \leq i, j \leq N$ having the ellipticity property

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2, \ \forall\, \xi \in \mathbb{R}^N$$

for some $\alpha > 0$. Let us consider the following elliptic problem

$$\begin{cases} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + g(x,u) = f(x) \quad in \quad \Omega \\ u = 0 \ \ on \ \ \partial\Omega \end{cases} \tag{2.12}$$

where the nonlinearity is given by the real valued Carathéodory function

$$g : \Omega \times \mathbb{R} \to \mathbb{R}.$$

The particular case when $g(x,u) = a_0(x)u$, with $a_0 \in C(\overline{\Omega}),\ a_0 > p > 0$ is studied in [11], p. 51 by using Lax-Milgram theorem.

Corollary 2.1. *. If $g(x,u)$ has bounded partial derivatives in u,*

$$m \leq \frac{\partial g}{\partial u} \leq M \quad in\ \Omega,\ (m, M > 0). \tag{2.13}$$

then problem (2.12) has an unique weak solution for every $f \in L^2(\Omega)$.

Proof. Indeed, we can apply Theorem 1.1 for the following functional background:

$$H = L^2(\Omega), \quad Au := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right), \quad D(A) := H^2(\Omega) \cap H_0^1(\Omega),$$

$F(u) := g(\cdot, u) - f.$

Then A is monotone:

$$(Au, u) = \int_\Omega \sum_{i,j=1}^{N} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i} \geq 0$$

and $I + A$ is surjective ([9], p. 177) and thus A is maximal monotone. The conditions (i) and (ii) follow from (2.13). □

(a2) In [5] it is studied the perturbed Laplace problem

$$\begin{cases} -\Delta u + Pu = f \quad in \quad \Omega \\ \quad u = 0 \quad on \quad \partial\Omega \end{cases} \tag{2.14}$$

using the variational theorem of Langenbach. We can apply Theorem 1.1, asking that $P : L^2(\Omega) \to L^2(\Omega)$ satisfies (i) and (ii).

Corollary 2.2. *If P is Gâteaux differentiable and*

$$m \cdot |h|^2 \leq < (DP)(u)h, h > \leq M \cdot |h|^2, \quad (m, M > 0)$$

then (2.14) has a unique solution.

Proof. Indeed, A is maximal monotone, where

$$Au := -\Delta u,\ D(A) := -H^2(\Omega) \cap H_0^1(\Omega).$$

□

In particular, we can apply this result to the Dirichlet problem for the stationary diffusion equation ([13])

$$-\Delta u(x) + au(x) + g(x, u(x)) = f(x), x \in \Omega,$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ has a bounded partial derivative

$$\left|\frac{\partial g}{\partial u}\right| \leq M \text{ in } \Omega$$

and $a > M$.

(a3) Periodic solutions of semilinear wave equation

Corollary 2.3. *Suppose that $A : D(A) \subset H \to H$ is maximal monotone and*

$$F \in C(\mathbb{R} \times H, H)$$

is $T-$periodic, i.e.

$$F(t + T, \cdot) = F(t, \cdot), \quad \forall\, t \in \mathbb{R}.$$

Then there exist $T-$periodic solutions for the semilinear abstract equation:

$$\begin{cases} -u'' + Au + F(t, u) = 0\,, \quad t \in \mathbb{R} \\ u(0) = u(T)\,,\ u'(0) = u'(T). \end{cases} \tag{2.15}$$

Proof. Let $H := L^2((0,T); H)$ and $Lu := -u'' + Au$, with $D(L) := \{u \in C^2([0,T]; H) \cap L^2((0,T), D(A)) \mid u(0) = u(T)\,,\ u'(0) = u'(T)\}$. Then L is maximal monotone and if F satisfies (i) and (ii), in particular a condition of type (2.13), then problem (2.15) has exactly one periodic solution. □

For example, we can apply this result to the periodic problem

$$\begin{cases} -u''(t) + au(t) - b \sin u(t) = f(t), t \in (0, T) \\ u(0) = u(T) = 0, \end{cases}$$

where $a > b > 0$ (see [13]).

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