Monotone semilinear equations in Hilbert spaces and applications

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Abstract.

Consider a abstract semilinear equation of the form \( Au + F(u) = 0 \), where \( A \) is a maximal monotone map acting into a real Hilbert space \( H \), and \( F \) is a Lipschitz strongly monotone map on \( H \). Such equations were studied by Amann [1] and Bartsch [2] in the case when \( A \) is a linear self-adjoint map with the spectrum \( \rho(A) \) and \( F \) is a Gâteaux differentiable map such that there exist real numbers \( \nu < \mu \) such that \([\nu, \mu] \subset \rho(A)\) and

\[
\nu \leq \frac{F(u) - F(v), u - v}{|u - v|^2} \leq \mu, \quad \forall u, v \in H, u \neq v.
\]

We have the following Theorem 1.1. Assume that \( A : D(A) \subset H \rightarrow H \) is maximal monotone and there exist \( m, M > 0 \) such that

(i) \( F(u) - F(v), u - v \geq m \cdot |u - v|^2 \), \( \forall u, v \in H \);
(ii) \( |F(u) - F(v)| \leq M \cdot |u - v| \), \( \forall u, v \in H \).

Then equation (1.1) has a unique solution.

Proof. We will show that there exists \( \lambda > 0 \) such that \( S_\lambda : H \rightarrow H, S_\lambda(u) := u - \lambda F(u) \) is a contraction. Indeed,

\[
|S_\lambda(u) - S_\lambda(v)|^2 = |u - v|^2 - 2\lambda < F(u) - F(v), u - v > + \lambda^2 |F(u) - F(v)|^2 \leq (1 - 2\lambda m + \lambda^2 M) |u - v|^2,
\]

thus

\[
|S_\lambda(u) - S_\lambda(v)| \leq c \cdot |u - v|
\]

with \( c := \sqrt{1 - 2\lambda m + \lambda^2 M} < 1 \), if \( \lambda \in (0, \frac{2m}{M}) \).

Now equation (1.1) can be written as

\[
(I + \lambda A)u - (u - \lambda F(u)) = 0,
\]

or

\[
(I + \lambda A)u = S_\lambda(u),
\]

where \( \lambda > 0 \) is taken as above. Using the fact that \((I + \lambda A)\) is invertible and \(|(I + \lambda A)^{-1}| \leq 1\) for each \( \lambda > 0 \) (because \( A \) is maximal monotone, e.g. [9], p.123), equation (1.6) is equivalent with

\[
u\lambda = (I + \lambda A)^{-1}S_\lambda(u).
\]

We have

\[
|(I + \lambda A)^{-1}S_\lambda(u) - (I + \lambda A)^{-1}S_\lambda(v)| = \leq |(I + \lambda A)^{-1}S_\lambda(u) - S_\lambda(v)| \leq c \cdot |u - v|, \quad u, v \in H.
\]

Therefore, \( u \mapsto (I + \lambda A)^{-1}S_\lambda(u) \) is a contraction having an unique fixed point, thus (1.7) and consequently (1.1) has an unique solution.

A similar result can be proved in the following:

Theorem 1.2. Suppose that \( F \) satisfy (i)+(ii) and \( A : D(A) \subset H \rightarrow H \) is bounded, compact and monotone. Then equation (1.1) has a unique solution.
Proof. Equation (1.1) can be equivalently written as
\[(\lambda I + A)u = T_\lambda(u),\] (1.8)
where \(T_\lambda(u) := \lambda u - F(u),\) \(\lambda > 0.\) We have
\[|T_\lambda(u) - T_\lambda(v)|^2 = \lambda^2|u - v|^2 - 2\lambda < F(u) - F(v), u - v > + |F(u) - F(v)|^2 \leq (\lambda^2 - 2\lambda m + M^2)|u - v|^2,
\]
therefore
\[|T_\lambda(u) - T_\lambda(v)| \leq \sqrt{\lambda^2 - 2\lambda m + M^2} \cdot |u - v|.\] (1.9)

Let us choose \(\lambda > \max\{\|A\|, \frac{M^2}{2m}\}\).
In particular, \(\lambda > \|A\|\) implies that \(\lambda I + A\) is invertible because
\[
\sigma(A) \subset [-\|A\|, \|A\|].
\]
Moreover,
\[|(\lambda I + A)u|^2 = \lambda^2|u|^2 + 2\lambda(Au, u) + |Au|^2 \geq \lambda^2|u|^2,
\]
(because \(A\) is monotone), or
\[|(\lambda I + A)u| \geq \lambda|u|,
\]
hence \(|(\lambda I + A)^{-1}| \leq \frac{1}{\lambda}\). Equation (1.8) is equivalent with
\[u = (\lambda I + A)^{-1}T_\lambda(u).\] (1.11)

We have
\[|(\lambda I + A)^{-1}T_\lambda(u) - (\lambda I + A)^{-1}T_\lambda(v)| = |(\lambda I + A)^{-1}(T_\lambda(u) - T_\lambda(v))| \leq \\
\leq \frac{1}{\lambda} \sqrt{\lambda^2 - 2\lambda m + M^2} \cdot |u - v|.
\]

Because \(\lambda > \frac{M^2}{2m}\), it results that
\[\gamma := \frac{1}{\lambda} \sqrt{\lambda^2 - 2\lambda m + M^2} < 1,
\]
therefore \(u \mapsto (\lambda I + A)^{-1}T_\lambda(u)\) is a contraction. Now equation (1.11) and consequently (1.1) has an unique solution. \(\Box\)

The operator \(A\) may be a nonlinear one, but when it is a linear operator we can obtain an existence result with nice applications (see [13]):

**Theorem 1.3.** Let \(A : D(A) \subseteq H \to H\) be a linear map and \(F : H \to H\) a nonlinear one with \(\text{Rg}(F) \subseteq \text{Rg}(A)\). Suppose that there exist two positive constants \(c > M\) such that:

i) \(A\) is strongly positive of constant \(c > 0\), i.e.,
\[\langle Ax, x \rangle \geq c \|x\|^2, \forall x \in D(A);\]

ii) \(F\) is Lipschitz continuous of constant \(M > 0\), i.e.,
\[|F(x) - F(y)| \leq M \|x - y\|, \forall x, y \in H.
\]

Then the uninhomogeneous equation
\[Au + F(u) = f\] (1.1’)
has a unique solution for each \(f \in \text{Rg}(A)\).

**Proof.** Indeed, i) implies that \(A^{-1}\) exists as a linear continuous map from \(H_1 := \text{Rg}(A)\) into \(H\) and
\[\|A^{-1}\|_{L(H_1, H)} \leq \frac{1}{c},
\]
Then, equation (1.1’) can be equivalently rewritten as
\[V(u) := (I + A^{-1}F)u = A^{-1}f =: g.\] (1.1’’)

It is easily seen that \(V\) is Lipschitz continuous, i.e.,
\[|V(x) - V(y)| \leq \alpha \|x - y\|, \alpha := 1 + \frac{M}{c}, \forall x, y \in H.
\]
and strongly monotone
\[\langle V(x) - V(y), x - y \rangle \geq \beta |x - y|^2, \beta := 1 - \frac{M}{c}, \forall x, y \in H.
\]
Thus \(S_\lambda u := u - \lambda(V(u) - g) = (I - \lambda V)u + \lambda g\) is a contraction
\[|S_\lambda x - S_\lambda y| \leq (1 - 2\lambda\beta + \alpha^2\lambda^2) |x - y|^2,\]
for \( \lambda \in \left( 0, \frac{2\beta}{\alpha^2} \right) \) and we can apply the contraction mapping theorem.

For the unique solution \( u^* \) of equation (1.1') we have the following estimation
\[
|u^*| = |A^{-1}(f - F(u^*))| \leq \frac{1}{C} |f - F(u^*)|
\]
where \( |F(u^*)| \leq |F(u^*) - F(0)| + |F(0)| \leq M |u^*| + |F(0)|. \) Consequently we have that
\[
|u^*| \leq \frac{1}{C - M} (|f| + |F(0)|).
\]
This inequality allows to prove the continuous dependence of the solution on the second term in (1.1'), that is,
\[
|u^*_1 - u^*_2| \leq \frac{1}{C - M} |f_1 - f_2|, \quad \forall f_1, f_2 \in Rg(A),
\]
where \( u^*_i \) denote the unique solutions of the equations
\[
Au + F(u) = f_i, \quad i = 1, 2,
\]
respectively. Therefore, under the hypothesis of Theorem 1.3, the problem (1.1'') is well posed.

2. APPLICATIONS TO DIFFERENTIAL EQUATIONS

(a1) Semilinear Elliptic Boundary Problems ([8])
Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain and \( a_{ij} \in C^1(\Omega) \), \( 1 \leq i, j \leq N \) having the ellipticity property
\[
\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \alpha |\xi|^2, \quad \forall \xi \in \mathbb{R}^N
\]
for some \( \alpha > 0 \). Let us consider the following elliptic problem
\[
\begin{cases}
- \sum_{i,j=1}^N \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + g(x,u) = f(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
(2.12)
where the nonlinearity is given by the real valued Carathéodory function
\[
g : \Omega \times \mathbb{R} \to \mathbb{R}
\]
The particular case when \( g(x,u) = a_0(x)u \), with \( a_0 \in C(\Omega) \), \( a_0 > p > 0 \) is studied in [11], p. 51 by using Lax-Milgram theorem.

Corollary 2.1. If \( g(x,u) \) has bounded partial derivatives in \( u \),
\[
m \leq \frac{\partial g}{\partial u} \leq M, \quad (m, M > 0).
\]
then problem (2.12) has an unique weak solution for every \( f \in L^2(\Omega) \).

Proof. Indeed, we can apply Theorem 1.1 for the following functional background:
\[
H = L^2(\Omega), \quad Au := - \sum_{i,j=1}^N \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right), \quad D(A) := H^2(\Omega) \cap H^1_0(\Omega),
\]
\[
F(u) := g(u) - f.
\]
Then \( A \) is monotone:
\[
(Au,u) = \int_{\Omega} \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} \geq 0
\]
and \( I + A \) is surjective ([9], p. 177) and thus \( A \) is maximal monotone. The conditions (i) and (ii) follow from (2.13). □

(a2) In [5] it is studied the perturbed Laplace problem
\[
\begin{cases}
-\Delta u + Pu = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
(2.14)
using the variational theorem of Langenbach. We can apply Theorem 1.1, asking that \( P : L^2(\Omega) \to L^2(\Omega) \) satisfies (i) and (ii).

Corollary 2.2. If \( P \) is Gâteaux differentiable and
\[
m |h|^2 \leq \langle DP(u)h, h \rangle \leq M |h|^2, \quad (m, M > 0)
\]
then (2.14) has a unique solution.

Proof. Indeed, \( A \) is maximal monotone, where
\[
Au := -\Delta u, \quad D(A) := -H^2(\Omega) \cap H^1_0(\Omega).
\]
□
In particular, we can apply this result to the Dirichlet problem for the stationary diffusion equation ([13])
\[-\Delta u(x) + au(x) + g(x, u(x)) = f(x), x \in \Omega,
\]
where \( g : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) has a bounded partial derivative
\[\left| \frac{\partial g}{\partial u} \right| \leq M \text{ in } \Omega \]
and \( a > M \).

(a3) Periodic solutions of semilinear wave equation

Corollary 2.3. Suppose that \( A : D(A) \subset H \rightarrow H \) is maximal monotone and \( F \in C(\mathbb{R} \times H, H) \) is \( T \)-periodic, i.e.
\[F(t + T, \cdot) = F(t, \cdot), \quad \forall t \in \mathbb{R}.
\]
Then there exist \( T \)-periodic solutions for the semilinear abstract equation:
\[
\begin{aligned}
-u'' + Au + F(t, u) &= 0, \quad t \in \mathbb{R} \\
u(0) &= u(T), \quad u'(0) = u'(T).
\end{aligned}
\]
(2.15)

Proof. Let \( H := L^2([0, T]; H) \) and \( Lu := -u'' + Au \), with \( D(L) := \{ u \in C^2([0, T]; H) \cap L^2((0, T), D(A)) \mid u(0) = u(T), \quad u'(0) = u'(T) \} \). Then \( L \) is maximal monotone and if \( F \) satisfies (i) and (ii), in particular a condition of type (2.13), then problem (2.15) has exactly one periodic solution. \( \square \)

For example, we can apply this result to the periodic problem
\[
\begin{aligned}
-u''(t) + au(t) - b \sin u(t) &= f(t), t \in (0, T) \\
u(0) &= u(T) = 0,
\end{aligned}
\]
where \( a > b > 0 \) (see [13]).

References