CREATIVE MATH. & INF.
17 (2008), No. 2, 61 - 69

Online version available at http://creative-mathematics.ubm.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Singularity of a boundary value problem of the elasticity equations in a polyhedron

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ABSTRACT.
In this work we are looking at the study of the regularity of a boundary value problem governed by the Lamé equations in a cylindrical domain. By studying the longitudinal displacement singularity along an edge and the perpendicular displacement singularity to the same edge, we arrive to describe the behavior of singular solutions of the *Lamé* equations in a polyhedron.

1. INTRODUCTION

Let Ω be a homogeneous, elastic and isotropic medium occupying a bounded domain in $\mathbb{R}^2$, limited by straight polygonal boundary Γ which is supposed to be regular, $\Gamma = \bigcup_{j=1}^{J} \Gamma_j$, $\Gamma_i \cap \Gamma_j = \varnothing$, $\forall i \neq j$, where $\Gamma_j =]S_j, S_{j+1}[$, and S_j are the different corners of Ω. $\omega_j, 0 < \omega_j \leq 2\pi$, $j = 0, ..., J$ represent the opening of the angle that makes Γ_j and Γ_{j+1} toward the interior of Ω, η^j and τ^j represent the unit outward normal vector and the tangent vector on Γ_j respectively.

L is the *Lamé* operator defined by:

$$Lu = \mu \Delta u + (\lambda + \mu) \nabla . div\, u$$

u, f represents the displacement vector, and external density force respectively. $\sigma(u)$ is the stress tensor given by *Hook*'s law using *Lamé* coefficients λ and μ ($\lambda > 0$ and $\lambda + \mu \geq 0$)

$$\sigma(u) = (\sigma_{ij}(u))_{ij}, \text{ where } \sigma_{ij}(u) = 2\mu\varepsilon_{ij}(u) + \lambda tr(\varepsilon(u))\delta_{ij}$$

where δ_{ij} is the *Kronecker* symbol and $\varepsilon_{ij}(u) = \frac{1}{2}(\partial_i x_j + \partial_j x_i)$ is the linearized tensor of deformation. We will suppose $\nu_0 = \frac{1}{2-\nu}$, where ν designate the *Poisson* coefficient such as $0 < \nu < \frac{1}{2}$.

In the case of a polyhedron, we consider a domain Q of $\mathbb{R}^3$, limited by straight polyhedral boundary Σ. It is considered a particular edge, denoted A, of Σ. It is assumed to fix ideas that A is carried by the axis $z'Oz$, the adjacent faces Γ_0 and Γ_ω are carried by the plans $\{y = 0\}$ and $\{y = ax\}$ respectively. The dihedral so definite has for measure ω toward the interior of Q.

It is indispensable to signal that the results that will be demonstrated in this work are not verified to the corners neighborhood. That's why, we fix an opened interval I , whose closure is interior to A. Besides we fix an neighborhood U of the origin O in $Q \cap \{z = 0\}$, such as $\overline{U} \times \overline{I}$ does not have any corners of Q. $\eta' = (\eta_1, \eta_2, \eta_3)^t = (\eta, \eta_3)^t$ and $\tau' = (\tau_1, \tau_2, \tau_3)^t = (\tau, \tau_3)^t$ represents the unit outward normal vector and the tangent vector on Σ respectively.

We consider the corresponding cylinder $Q = \Omega \times \mathbb{R}$ which has an edge along $z'Oz$.

For $f \in L^2(Q)^3$, the problem considered here consists of finding the displacement field $u : Q \longrightarrow \mathbb{R}^3$, if possible in $H^2(Q)^3$, satisfying:

$$(P) \begin{cases} Lu + f = 0 \text{ in } Q \\ \left(u.\eta', \left(\sigma(u).\eta'\right).\tau'\right) = 0 \text{ on } \Sigma \end{cases},$$

or equivalent variational form:

$$(P_V) \begin{cases} \text{Find } u \in V \text{ such as} \\ a(u, v) = \ell(v), \text{ for all } v \in V \end{cases}$$

where

$$\begin{aligned} a(u, v) &= \sum_{i,j=1}^{3} \int_Q \sigma_{ij}(u)\varepsilon_{ij}(v)dx, \ \ell(v) = \sum_{i=1}^{3} \int_Q f_i v_i dx, \\ V &= \left\{ v \in H^1(Q)^3 ; u.\eta = 0, \text{ in } \Sigma \right\}. \end{aligned}$$

It is assumed that u, therefore as f, is to bounded support in the direction of z.

To describe the behavior of u along an edge, it is necessary to introduce, as in P. Grisvard [5], the following three convolution kernels, in z:

Received: 24.03.2008; In revised form: 9.09.2008.; Accepted:
2000 *Mathematics Subject Classification.* 35Jxx.
Key words and phrases. *Edge, Fourier transform, Lamé system, singular behavior, singular function, transcendent equation.*

$$\begin{aligned} K_{\lambda,\mu,r}(r,z) &= \frac{r\sqrt{1+\nu}}{\pi\left[r^2+(1+\nu)z^2\right]} \\ K_{\lambda,\mu,\theta}(r,z) &= \frac{r}{\pi\left[r^2+z^2\right]} \\ K_{\lambda,\mu,z}(r,z) &= \frac{r\sqrt{1+\nu}}{\pi\left[(1+\nu)r^2+z^2\right]} \ . \end{aligned}$$

1.1. **Singular solutions in a polygon.** In B. Benabderrahmane [1] and P. Grisvard [8], found that the solutions of the problem (P) in a polygonal domain Q (in the case $f = 0$) are characterized by the following transcendent equation (1.1) :

$$\sin^2 \alpha\omega = \sin^2 \omega,\ \alpha \neq 0, \neq \pm 1 \tag{1.1}$$

where $\mathrm{Re}\alpha \in]0,1[$.

It is easy to verify that the solutions of the transcendent equation (1.1) are given by

$$\alpha_\ell = \frac{\ell\pi}{\omega} \pm 1,\ \ell \in \mathbb{N}^* \ .$$

Besides they are simple if $\omega \neq \frac{k\pi}{2}$, $k \in \mathbb{Z}^*$, else they are double. By the simple calculations we find that:

$*$ If $\omega < \frac{\pi}{2}$, then $u \in H^2(\Omega)^2$;

$*$ If $\omega = \frac{\pi}{2}, \pi$, it was a simple poles $\alpha = 0, \pm 1$;

$*$ If $\omega = \frac{3\pi}{2}$, then $\alpha = \frac{1}{3}$ is a double root.

In the other cases, there is only one simple real root when $\omega \in \left]\pi, \frac{3\pi}{2}\right[\bigcup \left]\frac{3\pi}{2}, 2\pi\right[$; and no solution when $\omega \in \left]\frac{\pi}{2}, \pi\right[$.

It is known in B. Benabderrahmane [2] that there are linearly independent functions S_α and $S'_\alpha \in V$, such as S_α, $S'_\alpha \notin H^2(\Omega)^2$ and LS_α, $LS'_\alpha \in L^2(\Omega)^2$ and as the *Lamé* operator is an isomorphism of

$$Sp\left(H^2(\Omega)^2, S_\alpha, S'_\alpha\right) \cap V \text{ on } L^2(\Omega)^2$$

where the Sp symbol designates the vector space generated by elements that are contained in parentheses that follow. These functions are given explicitly, in B. Benabderrahmane [2], by $S_\alpha(r,\theta) = r^\alpha \Psi_\alpha(\theta)$ such as

$$\Psi_\alpha(\theta) = \begin{cases} [(\rho_0-\rho_1)\sin(\alpha+1)\omega - 2\rho_1\sin(\alpha-1)\omega]\cos\alpha\theta + \\ \qquad (\rho_0+\rho_1)\sin(\alpha+1)\omega\cos(\alpha-2)\theta \\ [(-\rho_1+\rho_0)\sin(\alpha+1)\omega - 2\rho_1\sin(\alpha-1)\omega]\sin\alpha\theta \\ \qquad -(\rho_0+\rho_1)\sin(\alpha+1)\omega\sin(\alpha-2)\theta \end{cases} \tag{1.2}$$

where $\rho_0 = \nu_0(\alpha-1) - 2$, $\rho_0 = \nu_0(\alpha+1) + 2$.

2. Singularity in a polyhedron

The behavior of the singular solutions of *Lamé* equations in a polyhedron is described by the following theorem:

Theorem 2.1. *Let $\omega < 2\pi$, $u \in V$. For $f \in L^2(Q)^3$, there are functions $C_\alpha, C'_\alpha, C_{\alpha'}$ and $C'_{\alpha'}$ such as $C_\alpha, C'_\alpha \in H^{1-\alpha}(\mathbb{R})$, $C_{\alpha'}, C'_{\alpha'} \in H^{1-\alpha'}(\mathbb{R})$ verifying*

$$\left\{\begin{aligned} & u_r - \sum_{\alpha,\,0<Re\alpha<1} \left(K_{\lambda,\mu,r}(r,z) * C_\alpha\right) r^\alpha \Psi_{\alpha,r}(\theta) - \\ & \quad - \sum_{\alpha,\,0<Re\alpha<1} \left(K_{\lambda,\mu,r}(r,z) * C'_\alpha\right) r^\alpha \Phi_{\alpha,r}(\theta) \end{aligned}\right. \in H^2(U\times\mathbb{R}) \tag{1.3}$$

$$\left\{\begin{aligned} & u_\theta - \sum_{\alpha,\,0<Re\alpha<1} \left(K_{\lambda,\mu,\theta}(r,z) * C_\alpha\right) r^\alpha \Psi_{\alpha,\theta}(\theta) - \\ & \quad \sum_{\alpha,\,0<Re\alpha<1} \left(K_{\lambda,\mu,\theta}(r,z) * C'_\alpha\right) r^\alpha \Phi_{\alpha,\theta}(\theta) \end{aligned}\right. \in H^2(U\times\mathbb{R}) \tag{1.4}$$

$$\left\{\begin{aligned} & u_3 - \sum_{\alpha',\,0<Re\alpha'<1} \left(K_{\lambda,\mu,z}(r,z) * C_{\alpha'}\right) r^\alpha \Psi_{\alpha'}(\theta) - \\ & \quad \sum_{\alpha',\,0<Re\alpha'<1} \left(K_{\lambda,\mu,z}(r,z) * C'_{\alpha'}\right) r^\alpha \Phi_{\alpha'}(\theta) \end{aligned}\right. \in H^2(U\times\mathbb{R}) \tag{1.5}$$

where the functions

$$\Psi_\alpha(\theta) = \left(\Psi_{\alpha,r}(\theta), \Psi_{\alpha,\theta}(\theta)\right)$$

are given by (1.2) *and*

$$\Phi_\alpha = \frac{\partial \Psi_\alpha(\theta)}{\partial \alpha} = \left[\log r \Psi_\alpha(\theta) + \frac{\partial}{\partial r}\Psi_\alpha(\theta)\right] .$$

The functions $\Psi_{\alpha,r}(\theta), \Psi_{\alpha,\theta}(\theta)$ represents the radial part, angular part of $\Psi_\alpha(\theta)$ respectively. The functions $\Psi_{\alpha'}(\theta)$ are the first singular functions of the *Laplace* operator in a polygon.

The first sums in (1.3) and (1.4) are extended to all α; $\mathrm{Re}\alpha \in]0,1[$ simple roots of the equation (1.1), while the second sums are extended to all the double roots of the same equation. In (1.5), the first sums are extended to all $\alpha^{'}$ simple roots of the corresponding transcendent equation to the *Laplace* operator with the boundary conditions associated and the second sums are extended to all $\alpha^{'}$ double roots of the same equation.

The symbol $*$ represents the convolution in relation to z. The indices r, θ and z in the relations $(1.3), (1.4)$ and (1.5) are, respectively, the radial component, angular and longitudinal vector by using cylindrical coordinates.

For more details, we are going to give the similar of the Theorem 2.1, in the following cases:

- Case of simple roots such as $0 < \mathrm{Re}\alpha < 1$;
- Case of double roots such as $0 < \mathrm{Re}\alpha < 1$;
- Case of the fissure $(\omega = 2\pi)$.

Theorem 2.2. *We assume that* $\omega \in \left]\pi, \frac{3\pi}{2}\right[\bigcup \left]\frac{3\pi}{2}, 2\pi\right[$. *Let* $u \in V$ *be a variational solution, is to bounded support in the direction of* z. *For all* $f \in L^2(Q)^3$, *there are functions* C *and* C_α *such as*

$$C \in H^{1-\frac{\pi}{\omega}}(\mathbb{R}), C_\alpha \in H^{1-\alpha}(\mathbb{R}) \text{ and}$$

$$\begin{cases} u_r - \sum\limits_{\alpha,\, 0<\alpha<1} (K_{\lambda,\mu,r}(r,z) * C_\alpha) r^\alpha \Psi_{\alpha,r}(\theta) \in H^2(U \times \mathbb{R}) \\ u_\theta - \sum\limits_{\alpha,\, 0<\alpha<1} (K_{\lambda,\mu,\theta}(r,z) * C_\alpha) r^\alpha \Psi_{\alpha,\theta}(\theta) \in H^2(U \times \mathbb{R}) \\ u_3 - (K_{\lambda,\mu,z}(r,z) * C) r^{\frac{\pi}{\omega}} \cos\left(\frac{\pi}{\omega}\theta\right) \in H^2(U \times \mathbb{R}) \end{cases}$$

where $\alpha = \frac{\ell\pi}{\omega} \pm 1, \ell \in \mathbb{N}^*$ *are the simple roots of the equation* (1.1).

For $\omega = \frac{3\pi}{2}$, $\alpha = \frac{1}{3}$ is a double root of the equation (1.1). Therefore, it is necessary to modify the result of the Theorem 2.2 as follows: there are two constants C and $C^{'}$ such as

$$C \in H^{\frac{2}{3}}(\mathbb{R}), C^{'} \in H^{\frac{1}{3}}(\mathbb{R}) \text{ and}$$

$$\begin{cases} u_r - (K_{\lambda,\mu,r}(r,z) * C) r^{\frac{1}{3}} \Phi_{\frac{1}{3},r}(\theta) \in H^2(U \times \mathbb{R}) \\ u_\theta - (K_{\lambda,\mu,\theta}(r,z) * C) r^{\frac{1}{3}} \Phi_{\frac{1}{3},\theta}(\theta) \in H^2(U \times \mathbb{R}) \\ u_3 - \left(K_{\lambda,\mu,z}(r,z) * C^{'}\right) r^{\frac{2}{3}} \cos\left(\frac{2\theta}{3}\right) \in H^2(U \times \mathbb{R}) \end{cases}.$$

In the case $\omega = 2\pi$, we obtain the existence of the functions C and $C^{'}$ of $H^{\frac{1}{2}}(\mathbb{R})$ such as

$$\begin{cases} u_r - (K_{\lambda,\mu,r}(r,z) * C) \sqrt{r} \Phi_{\frac{1}{2},r}(\theta) \in H^2(U \times \mathbb{R}) \\ u_\theta - (K_{\lambda,\mu,\theta}(r,z) * C) \sqrt{r} \Phi_{\frac{1}{2},\theta}(\theta) \in H^2(U \times \mathbb{R}) \\ u_3 - \left(K_{\lambda,\mu,z}(r,z) * C^{'}\right) \sqrt{r} \cos\left(\frac{\theta}{2}\right) \in H^2(U \times \mathbb{R}) \end{cases}.$$

The demonstration is essentially based on the study of the following points:

- Decompose every problem in plane part, u and u_θ, and in longitudinal part, u_3.
- Study of the longitudinal displacement singularity along an edge.
- Study of the perpendicular displacement singularity along an edge.

We begin by

2.1. **Problem decomposition.** We start by studying the *Lamé* solutions in the tridimensional domain $Q = \Omega \times \mathbb{R}$ that presents an edge along $z^{'}Oz$.

For $f \in L^2(Q)^3$, let $u \in V$ be a variational solution of (P), then we have

$$a(u,v) = \ell(v), \text{ where}$$

$$a(u,v) = \sum_{i,j=1}^{3} \int_Q \sigma_{ij}(u)\varepsilon_{ij}(v) dx_1 dx_2 dx_3$$

$$\ell(v) = \sum_{i=1}^{3} \int_Q f_i v_i dx_1 dx_2 dx_3 \ .$$

The invariance of the problems in relation to z implies the following partial regularity result:

Lemma 2.1. *We have*

$$\frac{\partial^2 u}{\partial x \partial z}, \frac{\partial^2 u}{\partial y \partial z} \text{ and } \frac{\partial^2 u}{\partial z^2} \in L^2(Q)^3 \ .$$

Let's decompose the fields u and f to the plane components and longitudinal component by posing:

$$u = (v, u_3)^t \text{ and } f = (g, f_3)^t$$

where v and g are vector fields of dimension 2 (also depend of z).

We will use the following notations:

- $\triangle_2$: *Laplace* in dimension 2 (variables x_1, x_2).
- ∇_2: *Gradient* in dimension 2 (variables x_1, x_2).
- Div_2: *Divergence* in dimension 2 (variables x_1, x_2).

Using these notations the *Lamé* equations in dimension 3 become

$$\begin{cases} \mu\left(\triangle_2 v + \dfrac{\partial^2 v}{\partial z^2}\right) + (\lambda+\mu)\nabla_2\left(Div_2 v + \dfrac{\partial u_3}{\partial z}\right) = g \\ \mu\left(\triangle_2 u_3 + \dfrac{\partial^2 u_3}{\partial z^2}\right) + (\lambda+\mu)\dfrac{\partial}{\partial z}\left(Div_2 v + \dfrac{\partial u_3}{\partial z}\right) = f_3 \ . \end{cases}$$

Thanks to Lemma 2.1, it can see that

$$\begin{cases} \mu\left(\triangle_2 v + \dfrac{\partial^2 v}{\partial z^2}\right) + (\lambda+\mu)\nabla_2 Div_2 v = g - (\lambda+\mu)\nabla_2\left(\dfrac{\partial u_3}{\partial z}\right) \in L^2(Q)^3 \\ \mu\left(\triangle_2 u_3 + \dfrac{\partial^2 u_3}{\partial z^2}\right) + (\lambda+\mu)\dfrac{\partial^2 u_3}{\partial z^2} = f_3 - (\lambda+\mu)\dfrac{\partial}{\partial z}(Div_2 v) \in L^2(Q)^3 \ . \end{cases} \tag{1.6}$$

This formulation has the advantage to decouple v and u_3. The the left member in the first equations in (1.6) concerns the plane components of u, while the right member concerns the longitudinal component.

2.2. **Study of the boundary conditions.** It is assumed that

$$\eta' = (\eta_1, \eta_2, \eta_3)^t = (\eta, \eta_3)^t \text{ and } \tau' = (\tau_1, \tau_2, \tau_3)^t = (\tau, \tau_3)^t \ .$$

The condition $u.\eta' = 0$ becomes $u_3\eta_3 = -v.\eta$. As $\eta_3 = 0$ and $\tau_3 = 1$ then

$$u.\eta' = 0 \Leftrightarrow v.\eta = 0 \text{ (no condition on } u_3) \ .$$

Concerning the condition on $\left(\sum(u).\eta'\right)$, we set $u = (v, 0) + (0, 0, u_3)$. Using the relations $\sigma_{ij}(u) = 2\mu\varepsilon_{ij}(u) + \lambda tr(\varepsilon(u))\delta_{ij}$, $i, j = 1, 2, 3$, it results

$$\sigma_{11}(v,0) = (\lambda+\mu)\frac{\partial u_1}{\partial x} + \lambda\frac{\partial u_2}{\partial y},\ \sigma_{12}(v,0) = \sigma_{21}(v,0) = \mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right),$$

$$\sigma_{13}(v,0) = \sigma_{31}(v,0) = \mu\frac{\partial u_1}{\partial z},\ \sigma_{23}(v,0) = \sigma_{32}(v,0) = \mu\frac{\partial u_2}{\partial z},$$

$$\sigma_{22}(v,0) = (\lambda+\mu)\frac{\partial u_2}{\partial y} + \lambda\frac{\partial u_1}{\partial x},\ \sigma_{33}(v,0) = \mu\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right),$$

$$\sigma_{11}(0,0,u_3) = \sigma_{22}(0,0,u_3) = \lambda\frac{\partial u_3}{\partial z}, \sigma_{12}(0,0,u_3) = \sigma_{21}(0,0,u_3) = 0,$$

$$\sigma_{13}(0,0,u_3) = \sigma_{31}(0,0,u_3) = \lambda\frac{\partial u_3}{\partial x},\ \sigma_{23}(0,0,u_3) = \sigma_{32}(0,0,u_3) = \lambda\frac{\partial u_3}{\partial y},$$

$$\sigma_{33}(0,0,u_3) = (\lambda+\mu)\frac{\partial u_3}{\partial z} \ .$$

Using the fact that $\eta_3 = 0$ and $\tau_3 = 1$, these last relations involve

$$\begin{cases} (\sigma(v,0).\eta).\tau = (\sigma(v).\eta).\tau + \mu\frac{\partial}{\partial z}(u_3\eta_3) = (\sigma(v).\eta).\tau \\ (\sigma(0,0,u_3).\eta).\tau = \lambda\dfrac{\partial u_3}{\partial \eta} \ . \end{cases}$$

Therefore, we have the conditions that must verify by each components of $u = (v, u_3)$ for the considered boundary conditions:

$$u.\eta' = 0 \Leftrightarrow v.\eta = 0 \text{ and no condition on } u_3$$

$$\left(\sum(u).\eta'\right).\tau' = 0 \Leftrightarrow \left(\sum(v).\eta\right).\tau = -\lambda\frac{\partial u_3}{\partial \eta} = 0 \ .$$

2.3. **Study of the longitudinal displacement along an edge.** In (1.6) the second equation is none other than the *Laplace* equation in Q, using a change of scale in z. By posing

$$z = \sqrt{\frac{\mu}{\lambda + 2\mu}} z'$$

we obtain

$$\mu \triangle_2 u_3 + (\lambda + 2\mu) \frac{\partial}{\partial z} \left(Div_2 v + \frac{\partial u_3}{\partial z} \right) = \mu \triangle_2 u_3 + \mu \frac{\partial^2 u_3}{\partial (z')^2} = \mu \triangle u_3 \ .$$

This result attached to the results of the preceding paragraph permits us, for the longitudinal displacement part, to deduce the following problem:

$$(P_1) \begin{cases} \Delta u_3 = f_3 \text{ in } Q \\ \dfrac{\partial u_3}{\partial \eta} = h \text{ on } \sigma \end{cases}$$

where $h \in H^{-\frac{1}{2}} (U \times \mathbb{R})$, thanks to Lemma 2.1.

The study of this problem is already made by P. Grisvard [10]. The application of results of P. Grisvard [9], concerning the *Laplace* equations, gives after change of scale in z the following decomposition of u_3:

$$u_3 - \sum_{\alpha,\, 0<\text{Re}\alpha<1} (K_{\lambda,\mu,z} (r, z) * C) r^{\alpha} \Psi_{\alpha} (\theta) \in H^2 (Q)$$

where $C \in H^{1-\alpha} (\mathbb{R})$ and the functions $\Psi_{\alpha} (\theta)$ are the first singular functions of the problem (P_1), which are given, see P. Grisvard [8], by $\Psi_{\alpha} (\theta) = \cos \alpha\theta$ where $K_{\lambda,\mu,z} (r, z)$ represents the kernel of the *Laplace* operator. This establishes the part of the Theorem 2.2 that concerns the longitudinal part u_3.

2.4. **Study of the perpendicular displacement singularity along an edge.** We analyze the behavior of v from the first equation of (1.6):

$$\mu \left(\triangle_2 v + \frac{\partial^2 v}{\partial z^2} \right) + (\lambda + \mu) \bigtriangledown_2 Div_2 v = g - (\lambda + \mu) \bigtriangledown_2 \left(\frac{\partial u_3}{\partial z} \right) \in L^2 (Q)^3 \ .$$

To simplify we note h the second member of this equation. Using the partial *Fourier* transformation in z, we see that the previous equation amounts to the following problem which is governed by the *Lamé* system resolving:

$$L\widehat{v} - \mu\zeta^2 \widehat{v} = \widehat{h} \ .$$

Concerning the boundary conditions, we can see that the conditions remain unaltered, we will be able to have the same conditions but non homogeneous. However by subtracting v to a field $u \in H^2 (Q)^2$ verifying the same conditions to limits that v, consequently the field $w = v - u$ verifies the homogeneous conditions. To simplify the notations, we will note this field again by v.

The uniqueness of the variational solution implies that $\widehat{v} \in D_L$ where

$$D_L = \left\{ u \in sp \left(H^2 (\Omega)^2, S_{\alpha}, S_{\alpha}' \right);\ \left(u.\eta', \left(\sigma(u).\eta' \right).\tau' \right) = 0, \text{ on } \Sigma \right\}$$

therefore

$$\widehat{v} = \widehat{v}_R + \sum_{\alpha,\, 0<\text{Re}\alpha<1} \widehat{C}_{\alpha} \Im_{\alpha}$$

where $\widehat{v}_R \in H^2 (Q)^2$ and $\widehat{C}_{\alpha} \in \mathbb{R}$, for all $\zeta \in \mathbb{R}$. Moreover, according B. Benabderrahmane [2], we have the following inequalities:

$$\begin{cases} \zeta^2 \|\widehat{v}_R\|_{L^2(Q)^2} + \zeta \|\widehat{v}_R\|_{H^1(Q)^2} + \|\widehat{v}_R\|_{H^2(Q)^2} \leq C \left\|\widehat{h}\right\|_{L^2(Q)^2} \\ \sum_{\alpha,\, 0<\text{Re}\alpha<1} \left|\widehat{C}_{\alpha}\right| |\zeta|^{1-\alpha} \leq C \left\|\widehat{h}\right\|_{L^2(Q)^2} \end{cases}$$

From where it comes that $\widehat{v} \in H^2 (Q)^2$ and $\widehat{C}_{\alpha} \in H^{1-\alpha} (\mathbb{R})$. Besides the following decomposition:

$$\widehat{v} = \widehat{v}_R + \sum_{\alpha,\, 0<\text{Re}\alpha<1} \widehat{C}_{\alpha} \Im_{\alpha}$$

which is equivalent by proceeding the inverse *Fourier* transformation, taking account the fact that $\widehat{f * g} = \widehat{f}.\widehat{g}$, to

$$\begin{cases} v_r = (v_R)_r + \sum_{\alpha,\, 0<\text{Re}\alpha<1} (K_{\lambda,\mu,r} (r, z) * C_{\alpha}) (S_{\alpha})_r \\ v_{\theta} = (v_R)_{\theta} + \sum_{\alpha,\, 0<\text{Re}\alpha<1} (K_{\lambda,\mu,\theta} (r, z) * C_{\alpha}) (S_{\alpha})_{\theta} \end{cases}$$

because

$$K_{\lambda,\mu,r}(r,\zeta) = e^{\frac{-r|\zeta|}{\sqrt{1+\nu}}} \text{ and } K_{\lambda,\mu,\theta}(r,\zeta) = e^{-r|\zeta|}$$

and by definition

$$(\Im_\alpha)_r = e^{\frac{-r|\zeta|}{\sqrt{1+\nu}}} (S_\alpha)_r \text{ and } (\Im_\alpha)_\theta = e^{-r|\zeta|} (S_\alpha)_\theta \quad .$$

This establishes the first two inclusions of the Theorem 2.2.

References

[1] Benabderrahmane, B. and Merouani, B., *Comportement singulier des solutions du système de Lamé dans un un polyèdre*, Rev. Roum. Sci. Tech. Méc. Appl., t. 44 (1999) No. 2, 231-239

[2] Benabderrahmane, B and Nouiri, B., *Résolvante du système de Lamé dans un polygone*, has been accepted for publication in Anal. Univ. Oradea, fasc. Matematică, Tom XV (2008), 219-238

[3] Grisvard, P., *Caractérisation de quelques espaces d'interpolation*, Arch. Rational Mech. Anal., 25, 1 (1967), 40-63

[4] Grisvard, P., *Alternative de Fredholm relative au problème de Derichlet dans un polygone ou un polyèdre*, Bollettino della U.M.I., 5 (1972), 132-164

[5] Grisvard, P., *Alternative de Fredholm rela tive au problème de Derichlet dans un polyèdre*, Annali Scuola Normale Superiore - Pisa, Calsse di Scienze, Série IV, Vol. II (1975), No. 3, 360-388

[6] Grisvard, P., *Résolvante du Laplacien dans un polygone et singularité des équations elliptiques ou paraboliques,* C.R.A.S, Paris, t. 301 Série I, No. 5 (1985)

[7] Grisvard, P., *Edge behavior of the solution of an elliptic problem*, Mathematics, No. 132 (1988), 281-299

[8] Grisvard, P., *Singularity in boundary value problems,* Masson, Paris Milan, Barcelone Bonn, 1992

[9] Grisvard, P., *Boundary value problems in plan polygone*, Instruction for use, E.D.P, Bulletin de la direction des études et recherches, Série C, Mathématiques, No. 1 (1986), 21-59

[10] Grisvard, P., *Singularité en élasticité*, Laboratoire de Mathématiques I.M.S.P. Parc valrose Nice France, 27 juin 1988

[11] Merouani, B., *Solutions singulières du système de l'élasticité dans un polygone pour différentes conditions aux limites*, Maghreb Maths. Rev., 5 (1996), No. 1-2, 95-112

[12] Nečas, J., *Les méthodes directes en théories des équations elliptiques*, Prague, 1967

[13] Lions, J. L. and Magenes, E., *Problèmes aux limites non homogènes et application*, Vol. I., Paris, Dunod, 1968

[14] Weidman, J., *Linear operators in Hilbert spaces*, Springer-Verlag, New York, Heirdelberg Berlinn Hattersheiman Main, 1976

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