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Shape preserving quadratic interpolation at Greville abscissae

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ABSTRACT.

The paper presents a method to construct a C^1 quadratic approximation function that combine the shape preserving properties of the variation diminishing spline function with the approximation properties of the interpolation function.

1. INTRODUCTION

There are many techniques to built a shape preserving interpolant. To achieve this we use the properties of the variation diminishing spline function.

We will approximate a given function $f : [a,b] \to \mathbb{R}$ with a quadratic spline function from the space $\mathcal{S}_{3,t} = \left\{\sum_{j=1}^{n} \alpha_j N_{j,3} : \alpha_j \in \mathbb{R},\ j = \overline{1,n}\right\}$, where $N_{j,3}(x) = (t_{j+3} - t_j)\left[t_j, ..., t_{j+3}; (\cdot - x)_+^2\right]$ are the quadratic B-spline functions corresponding to the nondecreasing knot sequence $(t_i)_{i=1}^{n+3} : t_1 = t_2 = t_3 = a < t_4 < ... < t_n < b = t_{n+1} = t_{n+2} = t_{n+3}$.

If we know the interpolation points $(x_i, y_i)_{i=1}^{n}$

$$f(x_i) = y_i,\ i = \overline{1,n} \tag{1.1}$$

then the n-vector $(\alpha_i)_{i=1}^{n}$ can be computed from the linear system

$$\sum_{j=1}^{n} \alpha_j N_{j,3}(x_i) = f(x_i),\ i = \overline{1,n}. \tag{1.2}$$

The system has exactly one solution if the collocation matrix $(N_{j,3}(x_i))_{i,j=1}^{n}$ is invertible.

Theorem 1.1. (Schoenberg-Whitney) *The matrix $(N_{j,3}(x_i))_{i,j=1}^{n}$ is invertible if and only if*

$$(N_{i,3}(x_i)) \neq 0,\ i = \overline{1,n},$$

i.e., if and only if

$$t_i < x_i < t_{i+3},\ \textit{all}\ i = \overline{1,n}. \tag{1.3}$$

The proof of this theorem can be found in [6].

If we choose the vector $(\alpha_i)_{i=1}^{n}$:

$$\alpha_i = f(t_i^*),\ i = \overline{1,n}, \tag{1.4}$$

with $(t_i^*)_{i=1}^{n}$ a sequence of average points given by

$$t_i^* = \frac{t_{i+1} + t_{i+2}}{2},\ i = \overline{1,n}, \tag{1.5}$$

then we obtain the so called variation diminishing spline

$$Vf(x) = \sum_{j=1}^{n} \alpha_j N_{j,3}(x),\ x \in [a,b], \tag{1.6}$$

whose shape preserving properties are often used in the Computer Design. The Vf approximant is not an interpolant but rather a smoothing approximant.

Our aim is to obtain a B-spline coefficient sequence $(\alpha_i)_{i=1}^{n}$ that fulfil both relations (1.2) and (1.4).

We study the case of monotone (increasing) interpolation data:

$$y_1 < y_2 < ... < y_n.$$

It is known that in generally for a given interpolation knots sequence $(x_i)_{i=1}^{n}$, it is impossible to find a sequence $(t_i)_{i=1}^{n+3}$ such that $x_i = \frac{t_{i+1}+t_{i+2}}{2}$, $i = \overline{1,n}$. We will expand the interpolation points and according to this we construct the demanded sequence of coefficients.

Theorem 1.2. *If the sequence of B-coefficients $(\alpha_i)_{i=1}^{n}$ is monotone then the quadratic spline Vf given in (1.6) is also monotone.*

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Proof. From (1.6) we have:

$$(Vf)'(x) = \sum_{j=1}^{n} \alpha_j N'_{j,3}(x).$$

By derivation the B-spline functions $N_{j,3}$

$$\begin{aligned} N'_{j,3}(x) &= (t_{j+3}-t_j)\left[t_j,...,t_{j+3};\left((\cdot - x)_+^2\right)'\right] = \\ &= 2(-1)\left(\left[t_{j+1},...,t_{j+3};(\cdot - x)_+\right] - \left[t_j,...,t_{j+2};(\cdot - x)_+\right]\right) \\ &= 2\left(\frac{N_{j,2}(x)}{t_{j+2}-t_j} - \frac{N_{j+1,2}(x)}{t_{j+3}-t_{j+1}}\right) \end{aligned}$$

we obtain

$$(Vf)'(x) = 2\alpha_1\left[t_1,t_2,t_3;(\cdot - x)_+\right] + \sum_{j=2}^{n} 2(\alpha_j - \alpha_{j-1})\frac{N_{j,2}(x)}{t_{j+2}-t_j} -$$
$$-2\alpha_n\left[t_{n+1},t_{n+2},t_{n+3};(\cdot - x)_+\right] = 2\sum_{j=2}^{n}(\alpha_j - \alpha_{j-1})\frac{N_{j,2}(x)}{t_{j+2}-t_j}.$$

If the function f is supposed to be monotone then the difference

$$\left(f\left(t_j^*\right) - f\left(t_{j-1}^*\right)\right) = (\alpha_j - \alpha_{j-1})$$

preserves the sign for all $j = \overline{2,n}$. It results that the function Vf has the same monotonicity as f. □

2. Knots selection

Let be a knots sequence $(t_i)_{i=1}^{2n+2}$ where each knot is of multiplicity two, excepting the terms at extremities which have multiplicity three:

$$t_1 = t_2 = t_3 < t_4 = t_5 < ... < t_{2n-2} = t_{2n-1} < t_{2n} = t_{2n+1} = t_{2n+2}. \tag{2.7}$$

More, we take this knots to be equal with the interpolation abscissae x_i :

$$t_{2i} = t_{2i+1} = x_i,\ i = \overline{1,n}.$$

Then we obtain for the average points (1.5):

$$t_{2i-1}^* = x_i, \tag{2.8}$$
$$t_{2i}^* = \frac{t_{2i+1}+t_{2i+2}}{2},\ \text{for all } i = \overline{1,n}. \tag{2.9}$$

So $(t_i^*)_{i=1}^{2n-1}$ is an extended sequence of the original abscissae $(x_i)_{i=1}^{n}$. More, this sequence $(t_i^*)_{i=1}^{2n-1}$ satisfy also the Schoenberg-Whitney condition:

$$t_i < t_i^* < t_{i+3},\ \text{for all } i = \overline{1,2n-1}. \tag{2.10}$$

We extend also the sequence of ordinate $(y_i)_{i=1}^{n}$ to $(y_i^*)_{i=1}^{2n-1}$:

$$y_{2i-1}^* := y_i, \quad i = \overline{1,n}, \tag{2.11}$$
$$y_{2i}^* := \frac{1}{2}y_{2i-1}^* + \frac{1}{2}y_{2i+1}^*,\ i = \overline{1,n-1}, \tag{2.12}$$

and we search a sequence $(\alpha_i^*)_{i=1}^{2n-1}$ such that the variation diminishing spline

$$Vf(x) = \sum_{j=1}^{2n-1}\alpha_j^* N_{j,3}(x), \tag{2.13}$$

with $\alpha_j^* = f\left(t_j^*\right)$, $j = \overline{1,2n-1}$, should become an interpolation function, that is

$$Vf(t_i^*) = y_i^*,\ i = \overline{1,2n-1}.$$

Lemma 2.1. *The collocation matrix $(N_{j,3}(t_i^*))_{i,j=1}^{2n-1}$, for the knots $(t_i)_{i=1}^{2n+2}$ given in (2.7) and the abscissae $(t_i^*)_{i=1}^{2n-1}$ given in (2.8), (2.9) has the following bounded form*

$$(N_{j,3}(t_i^*)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \dots & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.14}$$

and it has the inverse

$$\left(N_{j,3}\left(t_i^*\right)\right)^{-1}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ -0.5 & 2 & -0.5 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & -0.5 & 2 & -0.5 & \ldots & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}. \tag{2.15}$$

Proof. If $j=2k+1,\ k=\overline{1,n-1}$ then

$$\begin{aligned} N_{j,3}(x) &= \left(t_{j+3}-t_j\right)\left[t_j,...,t_{j+3};(\cdot-x)_+^2\right]= \\ &= \left(t_{2k+4}-t_{2k+1}\right)\left[t_{2k+1},t_{2k+2},t_{2k+3},t_{2k+4};(\cdot-x)_+^2\right]= \\ &= \left(t_{2k+4}-t_{2k+1}\right)\left[t_{2k+1},t_{2k+2},t_{2k+3},t_{2k+4};(\cdot-x)_+^2\right]= \\ &= \left(t_{2k+4}-t_{2k+1}\right)\left[t_{2k+1},t_{2k+2},t_{2k+3},t_{2k+4};(\cdot-x)_+^2\right]= \\ &= \left[t_{2k+2},t_{2k+3},t_{2k+4};(\cdot-x)_+^2\right]-\left[t_{2k+1},t_{2k+2},t_{2k+3};(\cdot-x)_+^2\right]= \\ &= \frac{1}{t_{2k+4}-t_{2k+2}}\left(\left[t_{2k+3},t_{2k+4};(\cdot-x)_+^2\right]-\left[t_{2k+2},t_{2k+3};(\cdot-x)_+^2\right]\right)- \\ &-\frac{1}{t_{2k+3}-t_{2k+1}}\left(\left[t_{2k+2},t_{2k+3};(\cdot-x)_+^2\right]-\left[t_{2k+1},t_{2k+2};(\cdot-x)_+^2\right]\right)= \\ &= \frac{1}{t_{2k+4}-t_{2k+2}}\left(\frac{\left(t_{2k+4}-x\right)_+^2-\left(t_{2k+3}-x\right)_+^2}{t_{2k+4}-t_{2k+3}}+2\left(t_{2k+2}-x\right)_+\right)- \\ &-\frac{1}{t_{2k+3}-t_{2k+1}}\left(-2\left(t_{2k+2}-x\right)_+-\frac{\left(t_{2k+2}-x\right)_+^2-\left(t_{2k+1}-x\right)_+^2}{t_{2k+2}-t_{2k+1}}\right), \end{aligned}$$

and from here for $x=t_{2k+1}^*=t_{2k+2}=t_{2k+3}$ we obtain

$$N_{2k+1,3}(t_{2k+1}^*)=\frac{1}{t_{2k+4}-t_{2k+2}}\left(\frac{\left(t_{2k+4}-t_{2k+1}^*\right)^2}{t_{2k+4}-t_{2k+3}}\right)=1.$$

Also for $x=t_{2k+2}^*=\dfrac{t_{2k+3}+t_{2k+4}}{2}$

$$N_{2k+1,3}(t_{2k+2}^*)=\frac{1}{t_{2k+4}-t_{2k+2}}\left(\frac{\left(t_{2k+4}-\frac{t_{2k+3}+t_{2k+4}}{2}\right)^2}{t_{2k+4}-t_{2k+3}}\right)=\left(\frac{1}{2}\right)^2=\frac{1}{4}.$$

In a similar way we obtain $N_{2k+1,3}(t_{2k}^*)=1/4$, $N_{2k,3}(t_{2k}^*)=1/2$, all other values are zero. □

Because the knots $(t_i^*)_{i=1}^{2n-1}$ satisfy the Schoenberg-Whitney condition (2.10), there is only one interpolation spline function; its coefficients will be

$$\left(\alpha_i^*\right)_i^T=\left(N_{j,3}\left(t_i^*\right)\right)_{i,j}^{-1}\cdot\left(y_j^*\right)_j^T. \tag{2.16}$$

Then it results

$$\alpha_i^*=\begin{cases} y_i^*, & i=1,3,...,2n-1 \\ -0.5y_{i-1}^*+2y_i^*-0.5y_{i+1}^*, & i=2,4,...,2n-2 \end{cases}. \tag{2.17}$$

If we use the relation (2.12) then $\alpha_i^*=y_i^*,$ for all $i=\overline{1,2n-1}$.

Theorem 2.3. *The quadratic variation diminishing spline function (2.13) is a monotone interpolant.*

Proof. As we have seen the coefficients $(\alpha_i^*)_{i=1}^{2n-1}$ are obtained from the linear system (2.16) so the Vf is an interpolation function. Because $\alpha_i^*=y_i^*,$ for all $i=\overline{1,2n-1}$, and the sequence $(y_i^*)_{i=1}^{2n-1}$ is monotone (increasing), according to Lemma 1.2, the function Vf is monotone. □

This interpolant is of class C^0 and that is due to the fact that the third order spline have knots of multiplicity two. To increase the order of smoothness to C^1 we split the interior knots $t_{2i}=t_{2i+1}\left(=x_i\right),\ i=\overline{2,n-1}$, and let them symmetrically back away from the value x_i, but not to far, because the sequence $(t_i)_{i=1}^{2n+2}$ have to be increasing (see Figure 1):

$$t_{2i}=x_i-\lambda_i\cdot d_i,\ t_{2i+1}=x_i+\lambda_i\cdot d_i,\ i=\overline{2,n-1}, \tag{2.18}$$

where

$$d_i=\min\left(x_i-x_{i-1},x_{i+1}-x_i\right) \tag{2.19}$$

and $\lambda = (\lambda_i)_{i=2}^{n-1}$, $\lambda_i \in \left(0, \frac{1}{2}\right)$.

Lemma 2.2 (Existence). *a) If $\lambda_i \to 0$ for all $i = \overline{2, n-1}$, then the collocation matrix $\left(N_{j,3}\left(t_i^*\right)\right)_{i,j=1}^{2n-1}$, where the knots $(t_i)_{i=1}^{2n+2}$ and the abscissae $(t_i^*)_{i=1}^{2n-1}$ are given in (2.18) and in (1.5) respectively, tends to the form (2.14).*

b) For each $i = \overline{2, n-1}$ there is a value $\lambda_i \in \left(0, \frac{1}{2}\right)$, such that the sequence $(\alpha_i^*)_{i=1}^{2n-1}$ given in (2.16) is monotone.

Proof. a) From the previous construction of the knots $(t_i)_{i=1}^{2n+2}$ given in (2.18) we have:

$$\lim_{\lambda_i \to 0} (t_{2i}) = \lim_{\lambda_i \to 0} (t_{2i+1}) = x_i \text{ for all } i = \overline{2, n-1},$$

so the sequence $(t_i)_{i=1}^{2n+2}$ becomes the knots sequence (2.7) and Lemma 2.1 holds.

b) At the margins of the interval $[x_1, x_n]$ we have

$$x_1 = t_1^* = t_1 = t_2 = t_3 \Rightarrow y_1 = \alpha_1^* = y_1^*$$
$$x_n = t_{2n-1}^* = t_{2n} = t_{2n+1} = t_{2n+2} \Rightarrow y_n = \alpha_{2n-1}^* = y_{2n-1}^*.$$

For the interior abscissae $t_2^*, ..., t_{2n-2}^*$

$$\lim \left(\alpha_{2j-1}^*\right) = y_{2j-1}^*,\ j = \overline{2, n-1},$$

and

$$\lim \left(\frac{1}{4}\alpha_{2j-1}^* + \frac{1}{2}\alpha_{2j}^* + \frac{1}{4}\alpha_{2j+1}^*\right) = y_{2j}^*,\ j = \overline{1, n-1},\ \text{if all } \lambda_i \to 0.$$

It results

$$\lim \left(\frac{1}{2}\alpha_{2j}^*\right) = y_{2j}^* - \frac{1}{4}\left(y_{2j-1}^* + y_{2j+1}^*\right) = \frac{1}{2} y_{2j}^*,\ j = \overline{1, n-1}.$$

It follows that for each y_j^*, $j = \overline{1, 2n-1}$ there exist a vicinity denoted by $V_{y_j^*}$, such that $\alpha_j^* \in V_{y_j^*}$. Because the sequence $(y_i^*)_{i=1}^{2n-1}$ is strictly increasing it follows that there are the values $\lambda_i \in \left(0, \frac{1}{2}\right)$, $i = \overline{2, n-1}$, such that the vicinities $V_{y_j^*}$, $j = \overline{1, 2n-1}$ become disjoint. For this values the sequence $(\alpha_i^*)_{i=1}^{2n-1}$ (2.16) becomes increasing. □

To obtain the values λ_i we will use an iteration method: we start with the initial value $\lambda_i = \frac{1}{3}$ for all $i = \overline{2, n-1}$, and decrease this value if it is necessary.

However, to obtain a monotone sequence $(\alpha_i^*)_{i=1}^{2n-1}$ we must not modify all the values λ_i, $i = \overline{2, n-1}$. We focus on how to remove a nonmonotone term of this sequence.

Suppose that $\alpha_{2p}^* = f\left(t_{2p}^*\right)$ is the first coefficient that does not respect the monotonicity of the sequence $(\alpha_i^*)_{i=1}^{2n-1}$, i.e. $\alpha_{2p}^* < \alpha_{2p-1}^*$.

In this case we will decrease the magnitudes of λ_p and λ_{p+1}. That affects only four knots $t_{2p}, t_{2p+1}, t_{2p+2}, t_{2p+3}$ and five abscissae: $t_{2p-2}^*, t_{2p-1}^*, t_{2p}^*, t_{2p+1}^*, t_{2p+2}^*$ (actually only three $t_{2p-2}^*, t_{2p}^*, t_{2p+2}^*$ the other two remains unchanged). This implies the change of seven B-spline functions: $N_{j,3}$, $j = \overline{2p-3, 2p+3}$. So we expect changes in seven columns of the collocation matrix.

Theorem 2.4. *a) The collocation matrix becomes:*

$$\left(N_{j,3}\left(t_i^*\right)\right) = \begin{array}{c} \begin{array}{ccccccccccc} & & 2p-3 & 2p-2 & 2p-1 & 2p & 2p+1 & 2p+2 & 2p+3 & & \end{array} \\ \left(\begin{array}{ccccccccccc} \ddots & & & & & & & & & & \\ \diamond & \diamond & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \diamond & \diamond & \diamond & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \diamond & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & \diamond & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \diamond & \diamond & \diamond \\ & & & & & & & & & & \ddots \end{array}\right) \end{array}$$

as $\lambda_p \to 0$ and $\lambda_{p+1} \to 0$. We are denoted with $*$ the nonzero values, and with $\diamond$ the values that does not changed during the iteration.

b) There are values $\lambda_p \in \left(0, \frac{1}{2}\right)$ and $\lambda_{p+1} \in \left(0, \frac{1}{2}\right)$ such that the coefficients α_{2p-1}^*, α_{2p}^*, α_{2p+1}^* fulfil the inequalities $\alpha_{2p-1}^* < \alpha_{2p}^* < \alpha_{2p+1}^*$.

Proof. a) If $\lambda_p \to 0$ and $\lambda_{p+1} \to 0$ then, t_{2p}, t_{2p+1} tend to x_p and t_{2p+2}, t_{2p+3} tend to x_{p+1}, so the value of the entries follows as in the Lemma 2.1.

b) From the interpolation conditions

$$\sum_{j=1}^{2n-1} \alpha_j^* N_{j,3}(t_i^*) = y_i^*,\ i = \overline{1, 2n-1} \tag{2.20}$$

when $i = 2p-1, 2p, 2p+1$, we obtain

$$\begin{aligned} \lim_{\lambda_p, \lambda_{p+1} \to 0} \left(\alpha_{2p-1}^*\right) &= y_{2p-1}^*, \\ \lim_{\lambda_p, \lambda_{p+1} \to 0} \left(\frac{1}{4}\alpha_{2p-1}^* + \frac{1}{2}\alpha_{2p}^* + \frac{1}{4}\alpha_{2p+1}^*\right) &= y_{2p}^*, \\ \lim_{\lambda_p, \lambda_{p+1} \to 0} \left(\alpha_{2p+1}^*\right) &= y_{2p+1}^*. \end{aligned} \tag{2.21}$$

Using (2.12), we obtain that

$$\lim_{\lambda_p, \lambda_{p+1} \to 0} \left(\frac{1}{2}\alpha_{2p}^* + \frac{1}{4}\left(\alpha_{2p-1}^* + \alpha_{2p+1}^*\right)\right) = y_{2p}^* = \frac{1}{2}\left(y_{2p-1}^* + y_{2p+1}^*\right),$$

so $lim_{\lambda_p, \lambda_{p+1} \to 0} \left(\alpha_{2p}^*\right) = \frac{1}{2}\left(y_{2p-1}^* + y_{2p+1}^*\right) = y_{2p}^*$.

It follows that there are values $\lambda_p \in \left(0, \frac{1}{2}\right)$ and $\lambda_{p+1} \in \left(0, \frac{1}{2}\right)$, such that each α_k^* fit to a vecinity of y_k^*, $k = 2p-1, 2p, 2p+1$ of radius $\delta/2$, where

$$\delta = \min\left(\left|y_{2p}^* - y_{2p-1}^*\right|, \left|y_{2p+1}^* - y_{2p}^*\right|\right).$$

Because the sequence $(y_i^*)_{i=1}^{2n-1}$ is strictly monotone we obtain the desired statement. □

The case $\alpha_{2p+1}^* < \alpha_{2p}^*$ can be treated in the same way. We use an iterative procedure to determine the demanded sequence $(\alpha_i^*)_{i=1}^{2n-1}$.

The pseudocode of the algorithm looks as follows.

Start with the initial values $\lambda_i = \frac{1}{3}$, *for all* $i = \overline{2, n-1}$.
Compute the knots and abscissae using the relations (2.18), (1.5).
Solve the linear system (2.20) where y_i^* *are given in (2.11) and (2.12).*
while $\alpha_{2p}^* < \alpha_{2p-1}^*$
  $\lambda_p = \frac{1}{2}\lambda_p$ *and* $\lambda_{p+1} = \frac{1}{2}\lambda_{p+1}$;
  recompute the values $t_{2p}, t_{2p+1}, t_{2p+2}, t_{2p+3}$ *and* $t_{2p-2}^*, t_{2p}^*, t_{2p+2}^*$;
  refresh the 7×7 submatrix of the collocation matrix and solve the linear system (2.20);
end.

As we have seen the ordinate y_i^* was chosen as the average of the original interpolation ordinates. To obtain a visually more pleasant approximation function we will take into consideration also the convexity of data. So we keep the relation (2.11), and instead of (2.12) we choose:

$$y_{2i}^* := \begin{cases} \left(2y_{2i-1}^* + y_{2i+1}^*\right)/3 & \text{if } \left(P_{i-1}, P_i, P_{i+1}\right), \left(P_i, P_{i+1}, P_{i+2}\right) \text{ are convex,} \\ \left(y_{2i-1}^* + 2y_{2i+1}^*\right)/3 & \text{if } \left(P_{i-1}, P_i, P_{i+1}\right), \left(P_i, P_{i+1}, P_{i+2}\right) \text{ are concave,} \\ \left(y_{2i-1}^* + y_{2i+1}^*\right)/2 & \text{else,} \end{cases} \tag{2.22}$$

$i = \overline{2, n-2}$, where $P_i = (x_i, y_i)$.

For the ordinate y_2^*, y_{2n-2}^*, we take:

$$y_2^* := \begin{cases} \left(2y_1^* + y_3^*\right)/3 & \text{if } \left(P_1, P_2, P_3\right) \text{ are convex,} \\ \left(y_1^* + y_3^*\right)/2 & \text{if } \left(P_1, P_2, P_3\right) \text{ are concave,} \end{cases} \tag{2.23}$$

and

$$y_{2n-2}^* := \begin{cases} \left(y_{2n-3}^* + y_{2n-1}^*\right)/2 & \text{if } \left(P_{n-2}, P_{n-1}, P_n\right) \text{ are convex,} \\ \left(y_{2n-3}^* + 2y_{2n-1}^*\right)/3 & \text{if } \left(P_{n-2}, P_{n-1}, P_n\right) \text{ are concave.} \end{cases} \tag{2.24}$$

Lemma 2.3. *If* $\left(P_{i-1}, P_i, P_{i+1}\right), \left(P_i, P_{i+1}, P_{i+2}\right)$ *are convex (concave) then the interpolation point* (t_{2i}^*, y_{2i}^*) *and* P_i, P_{i+1} *are also convex (concave),* $i = \overline{2, n-2}$, *and the conditions of Theorem 2.4 are satisfied.*

Proof. Assume that $\left(P_{i-1}, P_i, P_{i+1}\right), \left(P_i, P_{i+1}, P_{i+2}\right)$ are convex. Assume also that the initial values $\lambda_i := \frac{1}{3}$, $i = \overline{2, n-1}$. Then from (2.18) it results

$$\begin{aligned} x_i &< t_{2i+1} \le \frac{2}{3}x_i + \frac{1}{3}x_{i+1} \\ \frac{1}{3}x_i + \frac{2}{3}x_{i+1} &\le t_{2i+2} < x_{i+1} \end{aligned}$$

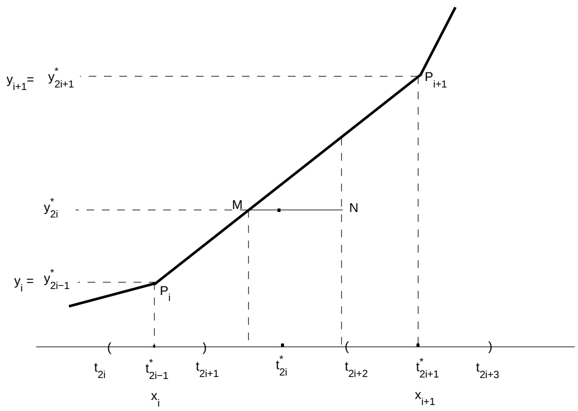

FIGURE 1.

so

$$\frac{2}{3}x_i + \frac{1}{3}x_{i+1} < \frac{t_{2i+1}+t_{2i+2}}{2} = t^*_{2i} < \frac{1}{3}x_i + \frac{2}{3}x_{i+1}.$$

From (2.22) it follows that $(t^*_{2i}, y^*_{2i}) \in |MN$, where $M = \left(1-\frac{1}{3}\right)P_i + \frac{1}{3}P_{i+1}$.

The concave case follows in the same way. Analogous idea is used for the interpolation points $(t^*_2, y^*_2), (t^*_{2n-2}, y^*_{2n-2})$.

It remains also to prove that for such a convex combination of $(y^*_i)_{i=1}^{2n-1}$ the sequence $(\alpha^*_i)_{i=1}^{2n-1}$ could be made monotone if we use the previous method.

From (2.21) we obtain in limit:

$$\lim_{\lambda_p,\lambda_{p+1}\to 0}\left(\frac{1}{2}\alpha^*_{2p} + \frac{1}{4}\left(y^*_{2p-1}+y^*_{2p+1}\right)\right)$$
$$= y^*_{2p} = (1-\beta)\,y^*_{2p-1} + \beta y^*_{2p+1},\ \beta \in [1/3, 2/3],$$

then

$$\lim_{\lambda_p,\lambda_{p+1}\to 0}\left(\alpha^*_{2p}\right) = (1-(2\beta-1/2))\,y^*_{2p-1} + (2\beta-1/2)\,y^*_{2p+1} \in \left(y^*_{2p-1}, y^*_{2p+1}\right).$$

So, there are disjoint vecinities U, V, W of

$$y^*_{2p-1}, (1-(2\beta-1/2))\,y^*_{2p-1} + (2\beta-1/2)\,y^*_{2p+1}, y^*_{2p+1}$$

such that: $\alpha^*_{2p-1} \in U$, $\alpha^*_{2p} \in V$, $\alpha^*_{2p+1} \in W$. Then results $\alpha^*_{2p-1} < \alpha^*_{2p} < \alpha^*_{2p+1}$. □

Theorem 2.5. *The following error bound hold for the interpolation function Vf*

$$||f - Vf|| = \max_{a\le x\le b}|f(x) - Vf(x)| \le 2\omega(f, ||t||),$$

where $||t|| = \max_i \Delta t_i$ is the meshsize of the knots $(t_i)_i$, and ω is the modulus of continuity.

Proof. If $\hat{x} \in [t_k, t_{k+1}]$ then

$$|f(\hat{x}) - Vf(\hat{x})| = \left| f(\hat{x}) - \sum_{j=k-2}^{k} f\left(t^*_j\right) N_{j,3}(\hat{x})\right| \le$$
$$\le \sum_{j=k-2}^{k}\left|f(\hat{x}) - f\left(t^*_j\right)\right| N_{j,3}(\hat{x}) \le$$
$$\le \max\{\left|f(\hat{x}) - f\left(t^*_j\right)\right| : k-2 \le j \le k\} \le$$
$$\le \max\{|f(u) - f(v)| : u, v \in \left[t^*_{k-2}, t_{j+1}\right] \text{ or } u, v \in [t_j, t^*_k]\} \le$$
$$\le \omega\left(f, \frac{3}{2}||t||\right) \le \left[1 + \frac{3}{2}\right]\omega(f, ||t||).$$

Then $||f - Vf|| \le 2\omega(f, ||t||)$. □

3. NUMERICAL EXAMPLES

We present the results of our algorithm on two data samples. First, we use the data from [4]:

x_i	-2	-1	-0.3	-0.2
y_i	0.25	1	$1/0.09$	25

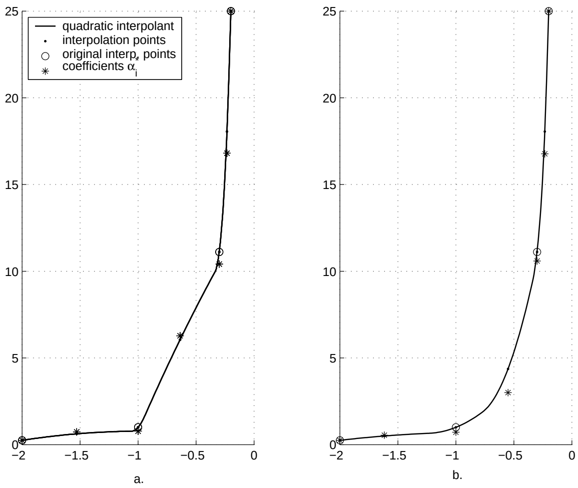

FIGURE 2.

where $y = f(x) = 1/x^2$.

In Figure 2a and 2b respectively we have used the average ordinate given in (2.11), (2.12) and (2.22), (2.23), (2.24). In the first case, we needed two iterations, $\lambda = \left(\frac{1}{3 \cdot 2^2}, \frac{1}{3 \cdot 2^2}\right)$ to obtain the monotone C^1 interpolant. In the second case there was no iteration, $\lambda = \left(\frac{1}{3}, \frac{1}{3}\right)$.

The interpolation function in Figure 2b is not convex, but, using the convex preserving properties of the variation diminishing spline function, for adequate interpolation ordinate, it could be built in the same way a convex interpolation function.

For the next data from [3],

x_i	7.99	8.09	8.19	8.7	9.2	10.	12.	15.	20.
y_i	0	$2.7642e-5$	$4.3749e-2$	0.16918	0.46942	0.9437	0.99863	0.999919	0.999994

that is much more restrictive, we have used for y^*_{2i} the relations (2.22), (2.23), (2.24).

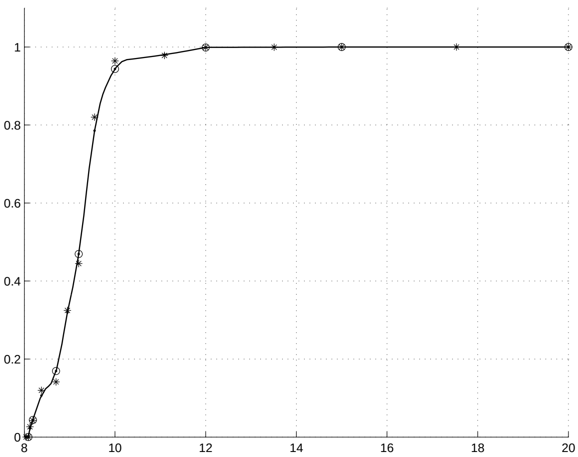

FIGURE 3.

The C^1 quadratic monotone interpolation function is represented in Figure 3 and the λ tension vector in this case is: $\lambda = \left(\frac{1}{3 \cdot 2^9}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3 \cdot 2^3}, \frac{1}{3 \cdot 2^2}\right)$.

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