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Statistical approximation properties of Kantorovich operators based on q-integers

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ABSTRACT.
In this paper we present two new generalizations of Kantorovich operators based on q-calculus. With the help of Bohman-Korovkin type theorem we obtain some statistical approximation properties for these operators. Also, by using the modulus of continuity, the statistical rate of convergence is established.

1. INTRODUCTION

In 1997 G. Phillips [14] proposed a generalization of the classical Bernstein polynomials based on q-integers. He estimated the rate of convergence and he obtained a Voronovskaya-type asymptotic formula for the new Bernstein operators. Then, in 2000 D. Bărbosu [2] constructed the bivariate q-Bernstein operators and established their approximation properties.

Recently, some new q-type generalizations of well known positive linear operators were introduced by several authors. For instance q-Meyer-König and Zeller operators were studied by T. Trif in [18], O. Doğru , V. Gupta, C. Orhan and O. Duman in [4], [6], [7]. Also, was established properties for the operators q-Bleimann, Butzer and Hahn [5] and q-Durrmeyer [3], [10]. On the other hand, the study of the statistical convergence for sequences of positive linear operators was attempted in the year 2002 by A.D. Gadjiev and C. Orhan (see [9]). In the present paper we introduce two q-extensions of Kantorovich operators and we investigate their statistical approximation properties. We also estimate the rate of statistical convergence for these new q-type operators. First of all, we recall the concept of statistical convergence. A sequence $(x_n)_n$ is said to be statistically convergent to a number L if for every $\varepsilon > 0$, $\delta\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0$, where $\delta(S) := \lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{j=1}^{N} \chi_S(j)$ is the natural density of the set $S \subseteq \mathbb{N}$. Here χ_S represents the characteristic function of S. We denote this limit by $st - \lim\limits_{n} x_n = L$ (see [9], [8], [16]).

2. CONSTRUCTION OF THE OPERATORS

There is no general definition of a "q-analogue". A q-analogue, also called q-extension or q-generalization of a mathematical object X is a family of objects $X(q)$, $q > 0$, (in most situations $q \in (0,1)$) such that $\lim\limits_{q\to 1} X(q) = X$. Sometimes q-generalization is not unique. In this study we present two q-extensions of the same operators. Before introducing the operators, we mention some definitions of q-calculus. For any fixed real number $q > 0$, we denote q-integers by $[k]_q$, where

$$[k]_q = \begin{cases} \dfrac{1-q^k}{1-q} & \text{if} \quad q \neq 1, \\ k & \text{if} \quad q = 1. \end{cases}$$

Note that $q \mapsto [k]_q$, $q > 0$, is a continuous function of q. The q-factorial is defined as follows

$$[k]_q! = \begin{cases} [1]_q \cdot [2]_q \cdot \ldots \cdot [k]_q & \text{if} \quad k = 1, 2, \ldots \\ 1 & \text{if} \quad k = 0, \end{cases}$$

and the q-binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}.$$

The q-binomial coefficients are also known as Gaussian polynomials (see [1] p. 35).

The q-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad \text{for any} \quad x \neq 0. \tag{2.1}$$

If $f'(0)$ exists, then D_q is extended by continuity at $x = 0$, i.e. $D_q f(0) = f'(0)$.

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It is easy to verify that

$$D_q x^n = \begin{cases} [n]_q\, x^{n-1} & \text{if} \quad n > 1, \\ 1 & \text{if} \quad n = 1 \end{cases} \tag{2.2}$$

and the product rule is

$$D_q\left(f(x)g(x)\right) = D_q f(x) g(qx) + f(x) D_q g(x). \tag{2.3}$$

The q-analogue of integration, introduced by Thomae [17] is given by

$$\int\limits_0^b f(x) d_q x = (1-q) \sum_{k=0}^{\infty} b q^k f\left(b q^k\right) \tag{2.4}$$

under the hypothesis that the series of the right hand side is convergent and over a general interval $[a,b]$

$$\int\limits_a^b f(x) d_q x = \int\limits_0^b f(x) d_q x - \int\limits_0^a f(x) d_q x. \tag{2.5}$$

If f is Riemann integrable on $[0,b]$, then $\int\limits_0^b f(x)dx = \lim\limits_{q\to 1^-} \int\limits_0^b f(x) d_q x$.

Moreover it is easy to check that q-integral defined by (2.4) is q-antiderivative, i.e.

$$D_q \int\limits_0^x f(t) d_q t = f(x), \quad x \neq 0. \tag{2.6}$$

Let $\alpha > 0$ be fixed. We denote by $\mathcal{T}_\alpha\left([0,b]\right)$ the set of all real-valued functions q-integrable on $[0,b]$ for any $q \in (0,\alpha)$. It is obvious that $\mathcal{T}_\alpha\left([0,b]\right)$ is a linear space.

We set $e_i, e_i(x) = x^i, i \geq 0$. In [14], Phillips defined q-Bernstein polynomials by

$$B_n\left(f;q;x\right) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \left(1-x\right)_q^{n-k} f\left(\frac{[k]_q}{[n]_q}\right), \tag{2.7}$$

where $(1-x)_q^{n-k} := \prod_{s=0}^{n-k-1}\left(1-q^s x\right)$ and proved the following identities.

$$B_n\left(e_0;q;x\right) = 1, \tag{2.8}$$

$$B_n\left(e_1;q;x\right) = x, \tag{2.9}$$

$$B_n\left(e_2;q;x\right) = x^2 + \frac{x(1-x)}{[n]_q}, \tag{2.10}$$

for all $n \in \mathbb{N}, q \in (0,1]$ and $x \in [0,1]$.

Now we are ready to introduce the first q-analogue of Kantorovich operators.

$$K_n\left(f;q;x\right) = [n+1]_q \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \left(1-x\right)_q^{n-k} \int_{q\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t) d_q t, \tag{2.11}$$

where $f \in \mathcal{T}_1\left([0,1]\right)$, $n \in \mathbb{N}$, $q \in (0,1)$. Based on (2.1), (2.6), (2.5) and taking into account the fact that

$$[k+1]_q = q\,[k]_q + 1, \tag{2.12}$$

it is obvious that K_n, $n \in \mathbb{N}$, are positive and linear operators. Furthermore, when $q \to 1^-$ the operators given by (2.11) reduce to the classical Kantorovich operators studied in [12].

Remark 2.1. *Using the characteristic function of the interval $\left(0, \frac{1}{[n+1]_q}\right]$, denoted by $\chi_{<n,q>}$, the operators K_n can be expressed as follows.*

$$K_n\left(f;q;x\right) = [n+1]_q \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \left(1-x\right)_q^{n-k} \int_0^1 f(t) \chi_{<n,q>}\left(t - q\frac{[k]_q}{[n+1]_q}\right) d_q t,$$

for all $n \in \mathbb{N}$.

3. A Bohman-Korovkin type theorem

Let D be a given interval of the real line. We denote by $C_B(D)$ the space of all functions f which are continuous on D and bounded on the entire line. Recall that $C_B(D)$ is a Banach space with respect to the sup-norm $\|\cdot\|$ given by

$$\|f\| := \sup_{x\in D} |f(x)|, \, f \in C_B(D).$$

We may assert that $C_B([0,1]) \subset \mathcal{T}_1([0,1])$ and the operators K_n, $n \in \mathbb{N}$, are well defined for any $f \in C_B([0,1])$. In [9] Gadjiev and Orhan proved the following Bohman-Korovkin type statistical approximation theorem.

Theorem A. *If the sequence of positive linear operators $A_n : C_B([a,b]) \to B([a,b])$ satisfies the conditions $st - \lim_n \|A_n e_i - e_i\| = 0$, for $i = 0,1,2$, then for any function $f \in C_B([a,b])$, we have $st - \lim_n \|A_n f - f\| = 0$, where $B([a,b])$ is the space of all real-valued functions bounded on $[a,b]$.*

To obtain our main result we need the next lemma.

Lemma 3.1. *For all $n \in \mathbb{N}$, $x \in [0,1]$ and for $0 < q < 1$, we have*

$$K_n(e_0;q;x) = 1, \tag{3.13}$$

$$K_n(e_1;q;x) = \frac{2[n]_q}{[2]_q[n+1]_q}qx + \frac{1}{[2]_q[n+1]_q}, \tag{3.14}$$

$$K_n(e_2;q;x) = \frac{3[n]_q[n-1]_q}{[3]_q[n+1]_q^2}q^3x^2 + \frac{3[n]_q}{[3]_q[n+1]_q^2}q(1+q)x + \frac{1}{[3]_q[n+1]_q^2}. \tag{3.15}$$

Proof. Let $q \in (0,1)$. Using the definition (2.11), the identities (2.2), (2.12) and (2.8) is easy to see that (3.13) holds true.

Taking into account (2.2), (2.12), (2.8) and (2.9), by direct computation, we obtain (3.14) as follows. For the sake of simplicity we denote by $X_{n,k,q} := x^k(1-x)_q^{n-k}$.

$$\begin{aligned}
K_n(e_1;q;x) &= \frac{[n+1]_q}{[2]_q}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q X_{n,k,q}\left(\frac{[k+1]_q^2}{[n+1]_q^2} - q^2\frac{[k]_q^2}{[n+1]_q^2}\right)\\
&= \frac{1}{[2]_q[n+1]_q}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q X_{n,k,q}\left([k+1]_q + q[k]_q\right)\\
&= \frac{1}{[2]_q[n+1]_q}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q X_{n,k,q}\left(2q[k]_q + 1\right)\\
&= \frac{2[n]_q}{[2]_q[n+1]_q}qB_n(e_1;q;x) + \frac{1}{[2]_q[n+1]_q}B_n(e_0;q;x)\\
&= \frac{2[n]_q}{[2]_q[n+1]_q}qx + \frac{1}{[2]_q[n+1]_q}.
\end{aligned}$$

A similar calculus reveals:

$$\begin{aligned}
K_n(e_2;q;x) &= \frac{[n+1]_q}{[3]_q}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q X_{n,k,q}\left(\frac{[k+1]_q^3}{[n+1]_q^3} - q^3\frac{[k]_q^3}{[n+1]_q^3}\right)\\
&= \frac{1}{[3]_q[n+1]_q^2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q X_{n,k,q}\left([k+1]_q^2 + q[k]_q[k+1]_q + q^2[k]_q^2\right)\\
&= \frac{1}{[3]_q[n+1]_q^2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q X_{n,k,q}\left(3q^2[k]_q^2 + 3q[k]_q + 1\right)\\
&= \frac{3[n]_q^2}{[3]_q[n+1]_q^2}q^2B_n(e_2;q;x) + \frac{3[n]_q}{[3]_q[n+1]_q^2}qB_n(e_1;q;x)\\
&+ \frac{1}{[3]_q[n+1]_q^2}B_n(e_0;q;x)\\
&= \frac{3[n]_q[n-1]_q}{[3]_q[n+1]_q^2}q^3x^2 + \frac{3[n]_q}{[3]_q[n+1]_q^2}q(1+q)x + \frac{1}{[3]_q[n+1]_q^2}.
\end{aligned}$$

□

Further on, we consider a sequence $(q_n)_n$, $q_n \in (0,1)$, such that

$$st - \lim_n q_n = 1. \tag{3.16}$$

We present the main result for the operators K_n given by (2.11).

Theorem 3.1. *Let $(q_n)_n$ be a sequence satisfying (3.16). Then, for all $f \in C_B([0,1])$, we have $st - \lim_n \|K_n(f; q_n; \cdot) - f\| = 0$.*

Proof. It is clear that

$$st - \lim_n \|K_n(e_0; q_n; \cdot) - e_0\| = 0. \tag{3.17}$$

From (3.14) and the relations

$$[n]_q \leq [n+1]_q, \quad [n]_q = 1 + q + q^2 + \ldots + q^{n-1}, \tag{3.18}$$

we get

$$\begin{aligned} |K_n(e_1; q_n; \cdot) - e_1| &\leq \left| \frac{2[n]_{q_n}}{[2]_{q_n}[n+1]_{q_n}} q_n - 1 \right| + \frac{1}{[2]_{q_n}[n+1]_{q_n}} \\ &= \frac{\left| \left(2 - [2]_{q_n}\right)[n]_{q_n} q_n - [2]_{q_n} \right|}{[2]_{q_n}[n+1]_{q_n}} + \frac{1}{[2]_{q_n}[n+1]_{q_n}} \\ &\leq \frac{q_n(1-q_n)[n]_{q_n}}{[2]_{q_n}[n+1]_{q_n}} + \frac{1+[2]_{q_n}}{[2]_{q_n}[n+1]_{q_n}}. \end{aligned}$$

Hence, we have

$$\|K_n(e_1; q_n; \cdot) - e_1\| \leq (1-q_n)q_n + \frac{2}{[n]_{q_n}} \leq 2\left((1-q_n)q_n + \frac{1}{[n]_{q_n}}\right). \tag{3.19}$$

Since $st - \lim_n q_n = 1$ we conclude $st - \lim_n \frac{1}{[n]_{q_n}} = 0$ and $st - \lim_n (1-q_n)q_n = 0$.

For a given $\varepsilon > 0$, let us define the following sets

$$A := \{n \in \mathbb{N} : \|K_n(e_1; q_n; \cdot) - e_1\| \geq \varepsilon\}, \ A_1 := \left\{n \in \mathbb{N} : (1-q_n)q_n \geq \frac{\varepsilon}{4}\right\},$$

$$A_2 := \left\{n \in \mathbb{N} : \frac{1}{[n]_{q_n}} \geq \frac{\varepsilon}{4}\right\}.$$

Based on (3.19) we have $\|K_n(e_1; q_n; \cdot) - e_1\| < 2\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) = \varepsilon$ for all $n \in \mathbb{N} \setminus (A_1 \cup A_2)$. This implies $A \subseteq A_1 \cup A_2$ and $\delta(A) \leq \delta(A_1) + \delta(A_2) = 0$.

Consequently, we get

$$st - \lim_n \|K_n(e_1; q_n; \cdot) - e_1\| = 0. \tag{3.20}$$

According to (3.15) and (3.18) we can write

$$\begin{aligned} |K_n(e_2; q_n; \cdot) - e_2| &\leq \left| \frac{3[n]_{q_n}[n-1]_{q_n}}{[3]_{q_n}[n+1]^2_{q_n}} q_n^3 - 1 \right| + \frac{3[n]_{q_n} q_n(1+q_n) + 1}{[3]_{q_n}[n+1]^2_{q_n}} \\ &< \frac{\left| 3[n]_{q_n}\left([n]_{q_n} - 1\right) q_n^2 - [3]_{q_n}[n]^2_{q_n} q_n^2 - 2[3]_{q_n}[n]_{q_n} q_n - [3]_{q_n} \right|}{[3]_{q_n}[n+1]^2_{q_n}} + \frac{3}{[n]_{q_n}} + \frac{1}{[n+1]^2_{q_n}} \\ &< \frac{[n]^2_{q_n} q_n^2 \left(3 - [3]_{q_n}\right)}{[3]_{q_n}[n+1]^2_{q_n}} + \frac{[n]_{q_n} q_n \left(3q_n + 2[3]_{q_n}\right)}{[3]_{q_n}[n+1]^2_{q_n}} + \frac{3}{[n]_{q_n}} + \frac{2}{[n+1]^2_{q_n}} \\ &< \left(3 - [3]_{q_n}\right) + \frac{8}{[n]_{q_n}} + \frac{2}{[n]^2_{q_n}}, \end{aligned}$$

which implies

$$\|K_n(e_2; q_n; \cdot) - e_2\| \leq 8\left(\left(3 - [3]_{q_n}\right) + \frac{1}{[n]_{q_n}} + \frac{1}{[n]^2_{q_n}}\right). \tag{3.21}$$

Since the sequence q_n satisfies (3.16) we have $st - \lim_n \left(3 - [3]_{q_n}\right) = 0$ and $st - \lim_n \frac{1}{[n]^2_{q_n}} = 0$. At this moment we define the following sets

$$B := \{n \in \mathbb{N} : \|K_n(e_2; q_n; \cdot) - e_2\| \geq \varepsilon\}, \ B_1 := \left\{n \in \mathbb{N} : \left(3 - [3]_{q_n}\right) \geq \frac{\varepsilon}{24}\right\},$$

$$B_2 := \left\{ n \in \mathbb{N} : \frac{1}{[n]_{q_n}} \geq \frac{\varepsilon}{24} \right\}, \qquad B_3 := \left\{ n \in \mathbb{N} : \frac{1}{[n]_{q_n}^2} \geq \frac{\varepsilon}{24} \right\}.$$

By using (3.21) we obtain $B \subseteq B_1 \cup B_2 \cup B_3$.

Hence, we have $\delta(B) \leq \delta(B_1) + \delta(B_2) + \delta(B_3) = 0$, which implies

$$st - \lim_n \|K_n(e_2; q_n; \cdot) - e_2\| = 0. \tag{3.22}$$

Finally, using (3.17), (3.20) and (3.22), the proof follows from Theorem A. □

Corollary 3.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $(q_n)_n$ be a sequence satisfying (3.16). For all $f \in C_B([0,1])$ one has*

$$st_A - \lim_n \|K_n(f; q_n; \cdot) - f\| = 0.$$

Replacing statistical convergence by uniform convergence we obtain the next result.

Corollary 3.2. *Let $(q_n)_n$ be a sequence satisfying $\lim_n q_n = 1$. For all $f \in C_B([0,1])$, we have $\lim_n \|K_n(f; q_n; \cdot) - f\| = 0$.*

4. Another q-extension of Kantorovich operators

We define the second q-generalization of Kantorovich operators as follows.

$$\mathcal{K}_n(f; q; x) = [n+1]_q \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \left(\frac{x}{q} \right)^k (1-x)_q^{n-k} \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t) d_q t, \tag{4.23}$$

for all $f \in \mathcal{T}_1([0,1])$, $n \in \mathbb{N}$, $q \in (0,1)$.

It is obvious that $\mathcal{K}_n$, $n \in \mathbb{N}$, are linear positive linear and converge to the classical Kantorovich operators for $q \to 1^-$.

Lemma 4.2. *For all $n \in \mathbb{N}$, $x \in [0,1]$ and $0 < q < 1$, we have*

$$\mathcal{K}_n(e_0; q; x) = 1, \tag{4.24}$$

$$\mathcal{K}_n(e_1; q; x) = \frac{[n]_q}{[n+1]_q} x + \frac{1}{[2]_q [n+1]_q}, \tag{4.25}$$

$$\begin{aligned} \mathcal{K}_n(e_2; q; x) = &\frac{[n]_q^2}{[n+1]_q^2} x^2 + \frac{[n]_q}{[n+1]_q^2} x(1-x) + \\ &+ \frac{(2q+1)}{[3]_q} \frac{[n]_q}{[n+1]_q^2} x + \frac{1}{[3]_q [n+1]_q^2}. \end{aligned} \tag{4.26}$$

Proof. By using (2.8), (2.9), (2.10), and the identities

$$[k+1]_q - [k]_q = q^k, \quad \int_a^b x^j d_q x = \frac{1}{[j+1]_q} \left(b^{j+1} - a^{j+1} \right),$$

(4.24)-(4.26) can be easily proven in a similar manner as (3.13)-(3.15). □

Theorem 4.2. *Let $(q_n)_n$ be a sequence satisfying (3.16). Then, for all $f \in C_B([0,1])$, we have $st - \lim_n \|\mathcal{K}_n(f; q_n; \cdot) - f\| = 0$.*

Proof. Using the same method as in Theorem 3.1, the proof follows by Lemma 4.2 and Theorem A. □

The relation between the derivative of classical Bernstein polynomials and Kantorovich operators is well-known (see [13] p. 30). In what follows we prove that a q-analogue of this relation exists between q-Bernstein operators (2.7) and q-generalization defined by (4.23).

In order to give our next result we need the following theorem ([11], p. 73).

Theorem B. *If F is any anti q-derivative of a function f, namely $D_q F = f$, continuous at $x = 0$, then $\int_0^a f(x) d_q x = F(a) - F(0)$.*

Theorem 4.3. *For any real function F continuous at $x = 0$, $n \in \mathbb{N}$, $q \in (0,1)$, we have*

$$D_q B_{n+1}(F; q; x) = \mathcal{K}_n(f; q; qx), \text{ where } f(x) = D_q F(x).$$

Proof. Using the equality $(1-x)_q^n = \sum\limits_{k=0}^{n} q^{k(k-1)/2} \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k$ (see [15], p. 336) and the product rule (2.3) in q-calculus we get

$$D_q (1-x)_q^{n-k+1} = -[n-k+1]_q (1-qx)_q^{n-k}.$$

From (2.7), (2.3) and the above relation we obtain

$$\begin{aligned} D_q B_{n+1}(F;q;x) &= \sum_{k=0}^{n+1} \left[\begin{array}{c} n+1 \\ k \end{array}\right]_q [k]_q x^{k-1} (1-qx)_q^{n-k+1} F\left(\frac{[k]_q}{[n+1]_q}\right) \\ &- \sum_{k=0}^{n+1} \left[\begin{array}{c} n+1 \\ k \end{array}\right]_q [n-k+1]_q x^k (1-qx)_q^{n-k} F\left(\frac{[k]_q}{[n+1]_q}\right) \\ &= [n+1]_q \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k (1-qx)_q^{n-k} F\left(\frac{[k+1]_q}{[n+1]_q}\right) \\ &- [n+1]_q \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k (1-qx)_q^{n-k} F\left(\frac{[k]_q}{[n+1]_q}\right). \end{aligned}$$

Based on Theorem B we can write

$$\begin{aligned} D_q B_{n+1}(F;q;x) &= [n+1]_q \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k (1-qx)_q^{n-k} \int_0^{\frac{[k+1]_q}{[n+1]_q}} f(t) d_q t \\ &- [n+1]_q \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k (1-qx)_q^{n-k} \int_0^{\frac{[k]_q}{[n+1]_q}} f(t) d_q t \\ &= [n+1]_q \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k (1-qx)_q^{n-k} \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t) d_q t \\ &= \mathcal{K}_n(f;q;qx). \end{aligned}$$

□

5. Rates of statistical convergence

In this section, using the modulus of continuity, we prove a theorem for the rate of statistical convergence of the operators $\mathcal{K}_n$ defined by (4.23).

Let D be a real interval and $f \in C(D)$. Then the modulus of continuity of f, denoted by $\omega_f(\delta)$ is defined as

$$\omega_f(\delta) := \sup\left\{\left|f\left(x'\right) - f\left(x''\right)\right| : x', x'' \in D, \left|x' - x''\right| \le \delta\right\}, \quad \delta > 0.$$

It is known that for a function $f \in C(D)$, we have $\lim\limits_{\delta \to 0^+} \omega_f(\delta) = 0$ and, for any $\delta > 0$,

$$\left|f\left(x'\right) - f\left(x''\right)\right| \le \left(\frac{\left|x' - x''\right|}{\delta} + 1\right) \omega_f(\delta). \tag{5.27}$$

Theorem 5.4. *Let $(q_n)_n$ be a sequence satisfying (3.16). For all $f \in C([0,1])$, we have $\|\mathcal{K}_n(f;q_n;\cdot) - f\| \le 2\omega_f(\delta_n)$, where*

$$\delta_n^2 = \left(1 - \frac{[n]_{q_n}}{[n+1]_{q_n}}\right)^2 + \frac{2[n]_{q_n} - 1}{[3]_{q_n} [n+1]_{q_n}^2}. \tag{5.28}$$

Proof. For all $f \in C([0,1])$ and $0 \le a < b \le 1$ we have $\left|\int\limits_a^b f(x) d_q x\right| \le \int\limits_a^b |f(x)| d_q x$. We denote by $Y_{n,k,q_n}(x) := \left[\begin{array}{c} n \\ k \end{array}\right]_{q_n} \left(\frac{x}{q_n}\right)^k (1-x)_{q_n}^{n-k}$. Using property (5.27), the above relation and Cauchy inequality for linear positive operators, we get

$$\begin{aligned} |\mathcal{K}_n(f;q_n;x) - f(x)| &= \left|[n+1]_{q_n} \sum_{k=0}^{n} Y_{n,k,q_n}(x) \int_{\frac{[k]_{q_n}}{[n+1]_{q_n}}}^{\frac{[k+1]_{q_n}}{[n+1]_{q_n}}} f(t) - f(x) d_{q_n} t\right| \\ &\le [n+1]_{q_n} \sum_{k=0}^{n} Y_{n,k,q_n}(x) \int_{\frac{[k]_{q_n}}{[n+1]_{q_n}}}^{\frac{[k+1]_{q_n}}{[n+1]_{q_n}}} |f(t) - f(x)| d_{q_n} t \end{aligned}$$

$$\begin{aligned} &\leq \left\{\frac{1}{\delta}\mathcal{K}_n\left(|t-x|\,;q_n;x\right)+1\right\}\omega_f\left(\delta\right) \\ &\leq \left\{\frac{1}{\delta}\left(\mathcal{K}_n\left((t-x)^2;q_n;x\right)\right)^{\frac{1}{2}}+1\right\}\omega_f\left(\delta\right). \end{aligned}$$

From (4.24)-(4.26) we obtain

$$\begin{aligned} \mathcal{K}_n\left((t-x)^2;q_n;x\right) &= \mathcal{K}_n\left(e_2;q_n;x\right)-2x\mathcal{K}_n\left(e_1;q_n;x\right)+x^2\mathcal{K}_n\left(e_0;q_n;x\right) \\ &\leq x^2\left(1-\frac{[n]_{q_n}}{[n+1]_{q_n}}\right)^2+x(1-x)\frac{[n]_{q_n}}{[n+1]_{q_n}^2} \\ &+ \frac{x}{[n+1]_{q_n}}\left(\frac{(2q_n+1)\,[n]_{q_n}}{[3]_{q_n}\,[n+1]_{q_n}}-\frac{2}{[2]_{q_n}}\right)+\frac{1}{[3]_{q_n}\,[n+1]_{q_n}^2} \\ &< \left(1-\frac{[n]_{q_n}}{[n+1]_{q_n}}\right)^2+\frac{[n]_{q_n}}{4\,[n+1]_{q_n}^2}+\frac{[n]_{q_n}-1}{[3]_{q_n}\,[n+1]_{q_n}^2} \\ &< \left(1-\frac{[n]_{q_n}}{[n+1]_{q_n}}\right)^2+\frac{2\,[n]_{q_n}-1}{[3]_{q_n}\,[n+1]_{q_n}^2}. \end{aligned}$$

The last inequality yields

$$\|\mathcal{K}_n\left(f;q_n;\cdot\right)-f\|\leq\left\{\frac{1}{\delta}\left[\left(1-\frac{[n]_{q_n}}{[n+1]_{q_n}}\right)^2+\frac{2\,[n]_{q_n}-1}{[3]_{q_n}\,[n+1]_{q_n}^2}\right]^{\frac{1}{2}}+1\right\}\omega_f\left(\delta\right).$$

Choosing $\delta:=\delta_n$ as in (5.28) the proof is finished. □

Since $(q_n)_n$ satisfies (3.16), the sequence $(\delta_n)_n$ is statistically null, which yields that $st-\lim\limits_n \omega_f\left(\delta_n\right)=0$. Therefore, Theorem 5.4 gives the rate of statistical convergence of $\mathcal{K}_n\left(f;q_n;\cdot\right)$ to f.

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