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Sensitivity analysis of the energy functional

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ABSTRACT.
In the present work a numerical method is developed for the determination of the $J \in (0, \infty)$ interval in the three critical points theorem. The critical surfaces are also approximated.

1. INTRODUCTION

Let us consider the following elliptic type partial differential equation with a boundary condition:

$$(P_\lambda) \begin{cases} -\triangle u = \lambda f(u), & x \in \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a compact set.

The problem (P_λ) is a simplified form of certain stationary waves in the nonlinear Schrödinger equation [8], where the potential energy is zero, and the nonlinear term f is a perturbation, which satisfies the conditions in the three critical points theorem (Theorem 1.1). Under these conditions (P_λ) is a resonant problem.

We assign an energy functional $E_\lambda : H_0^1(\Omega) \to \mathbb{R}$ to this problem defined on Sobolev space $H_0^1(\Omega)$ given by:

$$E_\lambda(u) = \frac{1}{2} \|u\|_{H_0^1}^2 - \lambda \int_\Omega F(u(x))\, dx$$

where

$$F(s) = \int_0^s f(x)\, dx.$$

We know that E_λ is a continuous derivable and the critical points of E_λ are the weak solutions to the problem (P_λ) [6].

The numerical calculation of the critical points is based on a special case of the three critical points theorem [7]:

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set, with a smooth boundary, and $f : \mathbb{R} \to \mathbb{R}$ a continuous function with $\sup_{x\in\mathbb{R}} \int_0^x f(t)\, dt > 0$. Assume that there are a, q, s, γ, with $q < \frac{n+2}{n-2}$ (if $n > 2$), $s < 2$ and $\gamma > 2$, such that*

$$|f(x)| \le a(1 + |x|^q) \ \forall x \in \mathbb{R},$$

$$\int_0^x f(t)\, dt \le a(1 + |x|^s) \ \forall x \in \mathbb{R}$$

and

$$\limsup_{x\to 0} \frac{\int_0^x f(t)\, dt}{|x|^\gamma} < +\infty.$$

Then there exists an open interval $J \subseteq [0, +\infty[$ such that for each $\lambda \in J$ the problem (P_λ) has at least three distinct weak solutions in $H_0^1(\Omega)$.

In the present work a numerical method is developed for the determination of the $J \subset (0, \infty)$ interval for concrete problems.

2. THE NUMERICAL METHOD

The basic idea of this method is to search the λ value for which the chosen u surface will be a critical surface for the E_λ energy functional.

The $E_\lambda(u)$ is a functional of type

$$E_\lambda(u) = \int_\Omega L(u(x), \nabla u(x))\, dx,$$

where L is the Lagrangian function of E_λ, defined as

$$L = \frac{1}{2} (\nabla u)^2 - \lambda F(u).$$

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Since Ω is a compact set and $u \in H_0^1(\Omega)$ it follows that L is of class C^1. Then there is a number $\delta > 0$ such that the composed function $L(v(x), \nabla v(x))$ is defined for all $x \in \Omega$ and for all $v \in C^1(\Omega, \mathbb{R})$ with $\|v-u\|_{C^1(\Omega)} < \delta$. Thus the integral

$$E_\lambda(v) = \int_\Omega L(v(x), \nabla v(x))\, dx$$

can be formed for all $v \in C^1(\Omega, \mathbb{R})$ with $\|v-u\|_{C^1(\Omega)} < \delta$. Consequently, the function

$$\Phi(t) := E_\lambda(u + t\varphi)$$

is defined for each $\varphi \in C^1(\Omega, \mathbb{R})$ and for $|t| < t_0$, where t_0 is some positive number less than $\delta / \|\varphi\|_{C^1(\Omega)}$. Moreover, Φ is of class C^1 on $(-t_0, t_0)$, whence the first variation $\delta E_\lambda(u, \varphi)$ of E_λ at u in direction φ is well defined by

$$\delta E_\lambda(u, \varphi) = \Phi'(0).$$

The critical points of the functional E_λ are the solutions of the relation

$$\delta E_\lambda(u, \varphi) = 0 \text{ for all } \varphi \in C_c^\infty(\Omega, \mathbb{R}),$$

which implies that

$$\delta E_\lambda(u, \varphi) = 0 \text{ for all } \varphi \in C_0^1(\Omega, \mathbb{R}). \tag{2.1}$$

Since L and u are of class C^1, it holds that

$$\delta E_\lambda(u, \varphi) = dE_\lambda(u, \varphi) = DE_\lambda(u)\varphi \text{ for all } \varphi \in C_0^1(\Omega, \mathbb{R}), \tag{2.2}$$

where $DE_\lambda(u)$ is the Fréchet derivative of E_λ at u and $dE_\lambda(u, \varphi)$ is the Gâteaux derivative of E_λ at u and direction φ. Thus, if u is a critical point of E_λ, by relations (2.1) and (2.2) we can conclude that

$$E_\lambda(u + t\varphi) - E_\lambda(u) = t dE_\lambda(u, \varphi) + o(t) = o(t) \text{ for all } \varphi \in C_0^1(\Omega, \mathbb{R}).$$

Moreover, for $t < |t_0|$ with sufficiently small value $t_0 = t_0(\varphi) > 0$ we have

$$E_\lambda(u + t\varphi) - E_\lambda(u) \cong 0 \text{ for all } \varphi \in C_0^1(\Omega, \mathbb{R}).$$

by resolving the previous equation, we obtain

$$\lambda = \frac{1}{2} \frac{\|u + t\varphi\|^2 - \|u\|^2}{\int_\Omega [F(u(x) + t\varphi(x)) - F(u(x))]\, dx} \tag{2.3}$$

By using formula (2.3), we calculate the value of λ where u is a critical surface for E_λ.
We consider a compact set Ω in $\mathbb{R}^2$, and a sublinear function $f = f(u)$.

The algorithm of determination of λ:

Step 1: We take a grid with step h on Ω. We consider the values of the surfaces in the intersections: $u(x_i, y_j) = u_{i,j}$

Step 2: We approximate u with cubic spline surfaces where $(x_i, y_j, u_{i,j})$ are the control points.

Step 3: We approximate $(\nabla u)_{i,j}$ in the interior of Ω by using the following formulas of approximations at second degree

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} \approx \frac{u_{i+1,j} - u_{i-1,j}}{2h}, \quad \left(\frac{\partial u}{\partial y}\right)_{i,j} \approx \frac{u_{i,j+1} - u_{i,j-1}}{2h},$$

and at the margin by using

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} \approx \frac{u_{i\pm1,j} \mp u_{i,j}}{\pm h}, \quad \left(\frac{\partial u}{\partial y}\right)_{i,j} \approx \frac{u_{i,j\pm1} \mp u_{i,j}}{\pm h}.$$

Step 4: We count out the integrates by using the trapezoid rule:

$$\|u\|_{H_0^1}^2 = \int_\Omega |\nabla u|^2\, dx, \text{ and } \int_\Omega F(u(x))\, dx.$$

Step 5: Let $u := u + t \cdot \varphi$.

Step 6: We apply steps 3 and 4 for a new value at u.

Step 7: We calculate the value of λ with formula (2.3).

With the help of this algorithm we receive the alteration of λ when we change the height of one control point of the spline surface.

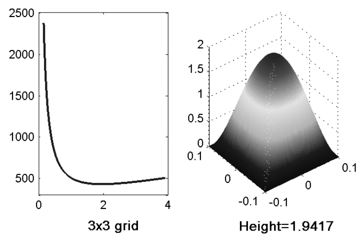


FIGURE 1. Graph of λ for 3×3 grid.

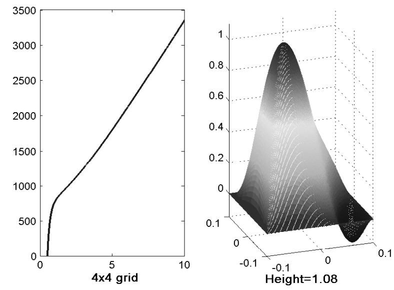


FIGURE 2. Graph of λ for 4×4 grid.

3. RESULTS

By using the Matlab program [3], we studied the variations of λ for the $f(u) = \arctan^2(u)$ sublinear function, where Ω is a square.

We take a 3×3 grid in Ω. The nodes of the grid are the control points of the surface. If we change the value of the control point in the middle of the grid between 0 and 4 and the values of the other control points remain the same, we obtain Figure 1.

The Ox axis indicates the height of the altering control point, and the Oy axis gives the values of λ. The graph gives the critical surface of the energy functional for a chosen λ.

The Figure 1 presents the surface where the graph has local minimum.

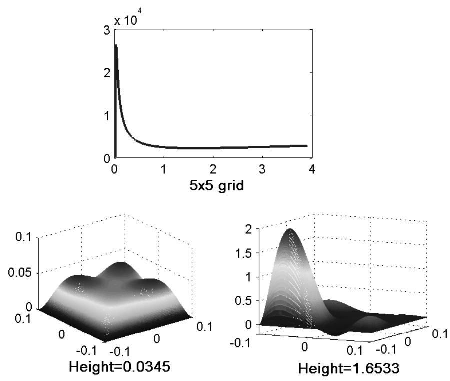


FIGURE 3. Graph of λ for 5×5 grid.

By considering four control points in the interior of Ω (4×4 grid), with three points fixed and the value of the forth point varying, we obtain Figure 2. The surface beside the graph is the surface where the graph has inflection point.

If we consider nine control points in the interior of Ω (5×5 grid), where the value in one of the extreme control points varies, and the values of the remaining eight control points are fixed, then we obtain Figure 3. The surfaces under the graph represent the surfaces where the graph has local maximum and local minimum, respectively.

Another possibility is to consider nine control points in the interior of Ω (5×5 grid), where we vary the height of the control point in the middle of the grid. In this case we obtain Figure 4. The surfaces under the graphs represent the surfaces where the graphs have vertical asymptote, local maximum and local minimum, respectively.

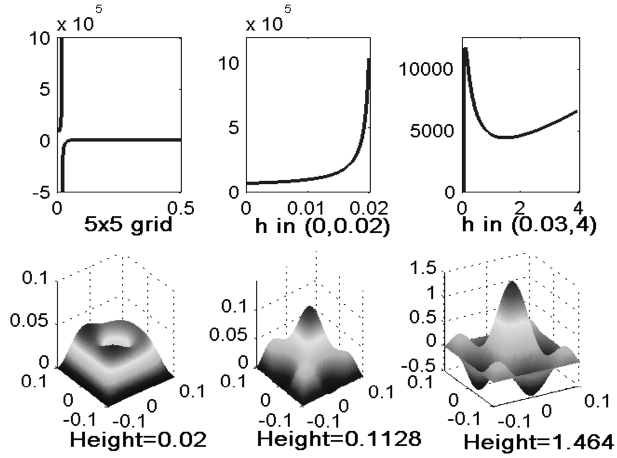


FIGURE 4. Graph of λ for 5×5 grid.

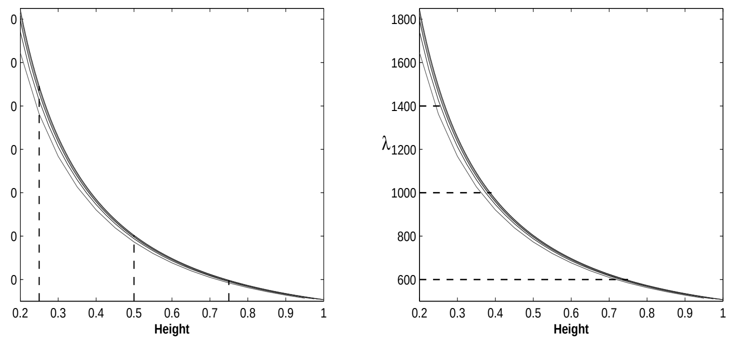


FIGURE 5. Convergence.

As approximation errors might intervine it is necessary to examine the correctness of this method. The method proves to be correct if we obtain better approximation results by taking lower steps in the derivation and integration formulas. We need to obtain a converging array of curves if the resolution is divided in two equal parts (see Figure 5).

The following two tables establish the convergence numerically. The first table shows the alteration of the distances between λ values when the resolution is redoubled. Accordingly, it is suggested to take small steps in the resolution where the little change of the control point height induces big changes of the λ value.

	0.25	**0.5**	**0.75**
0.05-0.025	63.31	15.33	6.06
0.025-0.0125	33.51	7.92	3.12
0.0125-0.00625	17.23	4.03	1.58
0.00625-0.003125	8.73	2.03	0.8
0.003125-0.0015625	4.39	1.02	0.4
0.0015625-0.00078125	2.21	0.51	0.2

The second table shows the alteration of the distances between the height of the control point when the resolution for a chosen λ value is redoubled.

	600	**1000**	**1400**
0.05-0.025	0.0118	0.0117	0.0121
0.025-0.0125	0.0061	0.0064	0.0061
0.0125-0.00625	0.0031	0.0031	0.0031
0.00625-0.003125	0.0016	0.0015	0.0016
0.003125-0.0015625	0.0007	0.0008	0.0007
0.0015625-0.00078125	0.0004	0.0004	0.0004

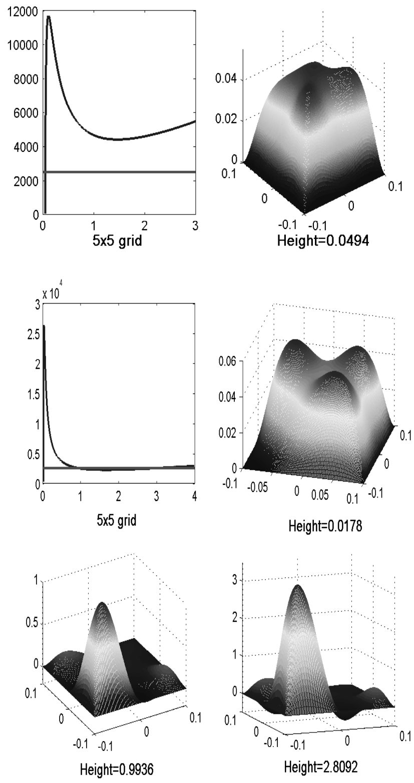


FIGURE 6. The possible critical surfaces for $\lambda = 2500$.

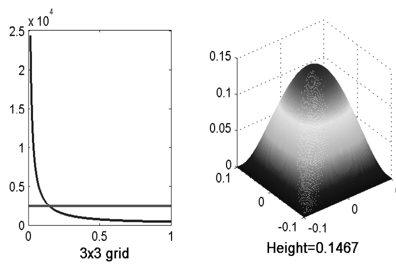
We also obtain a converging sequence, but here the received errors have the same order of magnitude, for different λ values. The obtained distances are halved by redoubling the resolution. The estimated error is $\varepsilon = 4 \cdot 10^{-4}$

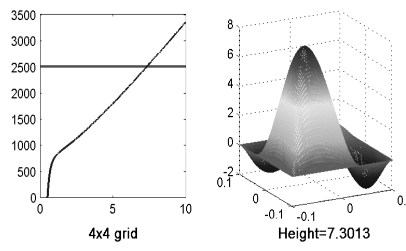
$$\lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon,$$

when the resolution is 0.00078125. The obtained figures are made with this resolution.

4. Conclusions

We can approximate the $J \subset (0, \infty)$ interval in the three critical points theorem with the presented method. By examining the obtained figures we can approximate the interval of λ with 0 and 26377.





We can also determine the values in the control points for which we find the critical surfaces of E_λ for a given λ. If, for example, we choose the value of λ equal to 2500 we obtain six surfaces which could be critical values of the energy functional E_λ (see Figure 6). The values of the altering control points height is specified under each figure.

5. Verification of the method

The obtained surfaces are only possible critical surfaces of the energy functional, also motivated by the fact that only one directional derivate has been taken in the calculations. We can check the validity of our statement with the help of the pdenonlin Matlab function from the Matlab partial differential equation toolbox [9]. Here we have to give as parameters the tolerance and the initial solution guess, which in our case will be the possible critical surface determinated by us.

We observe that every given possible critical surface will be a solution of the problem P_λ, under a certain *'tol'* tolerance. The level of the tolerance decreases if the value of *'tol'* grows. Figure 6 shows the decrease of the tolerance

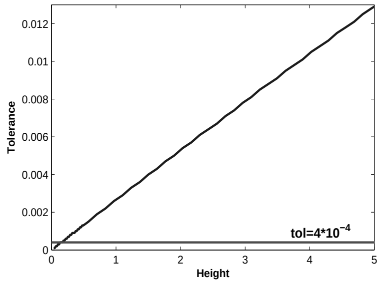


FIGURE 7. Tolerance

level when we raise the height of the control point, in the case of the surfaces presented by figure 3 (we take a 5×5 grid on Ω, and one of the extreme control points varies). This phenomenon is the consequence of the pdenonlin function derivatives scheme. The nonlinearities of the dependencies of the coefficients on the derivatives are not properly linearized by the scheme. When such nonlinearities are strong, the scheme reduces to the fix-point iteration and may converge slowly or not at all. We know that the nonlinearity of the surfaces increases when we grow the height of the control point. Consequently, it is hard to determine the tolerance level under which we can consider one surface as a solution.

The value of the tolerance is equal or greater than the calculation error given by us. Accordingly, we chose the tolerance value to be equal to $4 \cdot 10^{-4}$. Even so, the Matlab function accepts at least three surfaces determined by us as a solution to the P_λ problem in the $[433, 26377]$ interval of λ.

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