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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} *anniversary*

Generating the basins of attraction for Newton's method

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ABSTRACT. There are many ways to view the basins of attraction for Newton's method for complex polynomials. In the paper a program to perform this task is presented. It determines if the Newton's method is converging from any point in a rectangular domain in RxR, and computes the number of iterations necessary to attempt a root. The colors/number of iterations/roots and colors/roots diagrams are generated.

1. INTRODUCTION

1.1. **The basics.** The Newton's method is one of the most popular method to approximate a root for an equation $f(x) = 0$. This is an iterative method and the relation

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

is used to generate the approximates sequence and starting from the given value x_0.

Our work refers to the Newton's method for the complex coefficients polynomials roots and the attraction basins for this roots, too. A Pascal source program to determine the basins of attraction and some examples are presented.

1.2. **An algorithm to estimate the roots of a complex coefficients polynomial.** In this section we present an algorithm to estimate the roots of a complex coefficients polynomial using Newton's method.

Let us consider the complex coefficients polynomial:

$$f(z) = a_1 z^n + a_2 z^{n-1} + ... + a_n z + a_{n+1} \tag{1.1}$$

where $z = x + iy$ ($i^2 = -1$) is a complex variable.

Let u and v be the real functions on two variables such that:

$$f(z) = u(x,\ y) + iv(x,\ y),\ z = x + iy \tag{1.2}$$

$u = Re(f),\ v = Im(f)$ are real and imaginary parts of f respectively.

For $z = x + iy \in C$, we have

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$$z^n = (x+iy)^n = x_n + iy_n \tag{1.3}$$

where $x_1 = x,\ y_1 = y$. We have:

$$\begin{aligned} & z^n = (x+iy)^n = (x+iy)(x+iy)^{n-1} = \\ & \quad = (x+iy)(x_{n-1}+iy_{n-1}) = (xx_{n-1}-yy_{n-1}) + \\ & \quad + i(xy_{n-1}+yx_{n-1}) \end{aligned}$$

It results the following iterative relations to determine z^n:

$$\begin{cases} x_k = xx_{k-1} - yy_{k-1} & k = 1,2,...n \\ y_k = xy_{k-1} - yx_{k-1} & k = 1,2,...n \\ x_1 = x, y_1 = x, x_0 = 1,\ y_0 = 0 \end{cases} \tag{1.4}$$

and

$$\begin{aligned} & f(z) = a_1 z^n + a_2 z^{n-1} + ... + a_n z + a_{n+1} = \\ & = a_1(x_n + iy_n) + a_2(x_{n-1} + iy_{n-1}) + ... + \\ & + ... + a_n(x_1 + iy_1) + a_{n+1} = a_1 x_n + a_2 x_{n-1} + ... + \\ & + ... + a_n x_1 + a_{n+1} + i(a_1 y_n + a_2 y_{n-1} + ... + a_n y_1) \end{aligned}$$

From $f(z) = u(x,y) + iv(x,y)$ we can write:

$$\begin{aligned} u(x,y) &= a_1 x_n + a_2 x_{n-1} + ... + a_n x_1 + a_{n+1} \\ v(x,y) &= a_1 y_n + a_2 y_{n-1} + ... + a_n y_1 \end{aligned} \tag{1.5}$$

$$u(x,y) = \sum_{k=0}^{n} a_{n-k+1} x_k, \qquad v(x,y) = \sum_{k=1}^{n} a_{n-k+1} y_k \tag{1.6}$$

We compute the partial derivate of u and v:

$$\begin{aligned} \frac{\partial u}{\partial x}(x,y) &= \sum_{k=1}^{n} a_{n-k+1} \frac{\partial x_k}{\partial x}(x,y) \\ \frac{\partial v}{\partial x}(x,y) &= \sum_{k=1}^{n} a_{n-k+1} \frac{\partial y_k}{\partial y}(x,y) \end{aligned} \tag{1.7}$$

$$\begin{aligned} \frac{\partial u}{\partial y}(x,y) &= \sum_{k=1}^{n} a_{n-k+1} \frac{\partial x_k}{\partial y}(x,y) \\ \frac{\partial v}{\partial y}(x,y) &= \sum_{k=1}^{n} a_{n-k+1} \frac{\partial y_k}{\partial y}(x,y) \end{aligned} \tag{1.8}$$

We can prove that:

$$\begin{aligned} \frac{\partial x_k}{\partial x}(x,y) &= kx_{k-1} \\ \frac{\partial y_k}{\partial x}(x,y) &= ky_{k-1} \end{aligned} \tag{1.9}$$

$$\begin{aligned} \frac{\partial x_k}{\partial y}(x,y) &= -kx_{k-1} \\ \frac{\partial y_k}{\partial y}(x,y) &= ky_{k-1} \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} \frac{\partial u}{\partial x}(x,y) &= \sum_{k=1}^{n} ka_{n-k+1} x_{k-1} \\ \frac{\partial v}{\partial x}(x,y) &= \sum_{k=1}^{n} a_{n-k+1} y_{k-1} \end{aligned} \tag{1.11}$$

$$\begin{aligned} \frac{\partial u}{\partial y}(x,y) &= -\sum_{k=1}^{n} k a_{n-k+1} y_{k-1} \\ \frac{\partial v}{\partial y}(x,y) &= \sum_{k=1}^{n} k a_{n-k+1} x_{k-1} \end{aligned} \tag{1.12}$$

Let us consider the equation:

$$f(z) = 0,\ z = x + iy \tag{1.13}$$

Let $z = x + iy$ be a root of the equation (1.13).

Because $f(z) = u(x,y) + iv(x,y)$ the equation (1.13) becomes:

$$\begin{cases} u(x,y) = 0 \\ v(x,y) = 0 \end{cases} \tag{1.14}$$

so the equation (1.13) is equivalent to the system (1.14). We have:

$$x_k = x_{k-1} - \frac{u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} \tag{1.15}$$

$$y_k = y_{k-1} - \frac{u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} \tag{1.16}$$

where the functions $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are in $(x_{k-1},\ y_{k-1})$ and

$$x_k = x_{k-1} - \frac{u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}}{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2} \tag{1.17}$$

$$y_k = y_{k-1} - \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2} \tag{1.18}$$

The $x_k + iy_k$ sequence is converging to an approximative root of the equation $f(z) = 0$.

The iterative process (1.17) and (1.18) starts from $z_1 = x_1 + iy_1$ and stop at the condition

$$\frac{|x_{k+1} - x_k| + |y_{k+1} - y_k|}{|x_{k+1}| + |y_{k+1}|} < \varepsilon \tag{1.19}$$

where $\varepsilon > 0$. In the following we present the iterative relation to determine the values of $u, v, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ at a point $(x,\ y)$. For that, let us consider the polynomial function:

$$f(z) = a_1 z^n + a_2 z^{n-1} + ... + a_n z + a_{n+1}, \tag{1.20}$$

where $z = x + iy$. Let us denote

$$f(z) = u_{n+1}(x,y) + iv_{n+1}(x,y) = s_{n+1} \tag{1.21}$$

From (1.20) and (1.21) it results:

$$s_{n+1} = a_{n+1} + z(u_n + iv_n)$$

and $s_{n+1} = a_{n+1}$ + $(x + iy)(u_n + iv_n)$
and $s_{n+1} = a_{n+1} + (xu_n + yv_n) + i(yu_n + xv_n)$
and from (1.21) it results:

$$u_{n+1} = a_{n+1} + xu_n - yv_n$$

$$v_{n+1} = yu_n + xv_n \text{ with } u_1 = a_1, v_1 = 0 \tag{1.22}$$

relations to compute

$$f(z) = u_{n+1}(x, y) + iv_{n+1}(x, y).$$

The relations to compute $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ are determined in the following:

$$\begin{aligned} u_{n+1} &= a_{n+1} + (xu_n - yv_n) \\ v_{n+1} &= yu_n + xv_n \end{aligned} \tag{1.23}$$

By derivation we have:

$$\begin{aligned} \frac{\partial u_{n+1}}{\partial x} &= u_n + x\frac{\partial u_n}{\partial x} - y\frac{\partial v_n}{\partial x} \\ \frac{\partial v_{n+1}}{\partial x} &= v_n + y\frac{\partial u_n}{\partial x} + xy\frac{\partial v_n}{\partial x} \end{aligned} \tag{1.24}$$

2. THE BASINS OF ATTRACTION FOR NEWTON'S METHOD

Let us consider the polynomial

$$P(z) = a_1 z^n + a_2 z^{n-1} + ... + a_n z + a_{n+1}$$

where $z = x + iy(i^2 = -1)$ is a complex variable.

The ***Basin of Attraction*** for a root of P is the set of starting points for which the Newton's method is converging to this root.

3. THE PASCAL SOURCE PROGRAM FOR NEWTON'S METHOD

```
procedure NEWTON(l,j:integer;var k:byte;var sol:complex);
Begin
z.re:=alfa+l*h;;
z.im:=gama+j*q;
        Derivk(0,a,n,z,d);
        m:=modul(d);
        k := 0;
        While (m>=eps) and (k < Kmax) do
          begin
              k := k + 1;
             Derivk(1,a,n,z,d);
             if modul(d)¡1.0e-4000 then exit;
             Derivk(0,a,n,z,d);
             Derivk(1,a,n,z,d1);
             div(d,d1,d2);
             sub(z,d2,z);
             Derivk(0,a,n,z,d);
             m:=modul(d);
           end;
          k:=k mod 16;
          sol:=z;
End;{Procedure NEWTON}
```

```
unit Ucomplex;
interface
type complex=record
          re:extended;
          im:extended;
end;                      vcomplex=array[0..20] of complex;
procedure add(a,b:complex;var c:complex);
procedure sub(a,b:complex;var c:complex);
procedure mult(a,b:complex;var c:complex);
procedure inminco(i:integer;a:complex;var c:complex);
procedure div(a,b:complex;var c:complex);
function modul(a:complex):extended;
function permut(n,k:integer):integer;
        {Compute the product (n-k+1)....(n-1)n }
procedure Derivk (k:integer; a:v complex; n: integer; z : complex;
    var derk:complex);
    {Compute the k derivative in z, of a complex polynomial}
implementation

procedure add;
begin
c.re:=a.re+b.re;
c.im:=a.im+b.im;
end;

procedure sub;
begin
   c.re:=a.re-b.re;
   c.im:=a.im-b.im;
end;

procedure mult;
begin
  c.re:=b.re*a.re-a.im*b.im;
  c.im:=b.im*a.re+b.re*a.im;
end;

procedure inminco;
begin
   c.re:=i*a.re;
   c.im:=i*a.im;
end;

procedure div;
begin
c.re:=(a.re*b.re+a.im*b.im)/(sqr(b.re)+sqr(b.im));
c.im:=(a.im*b.re-a.re*b.im)/(sqr(b.re)+sqr(b.im));
end;
```

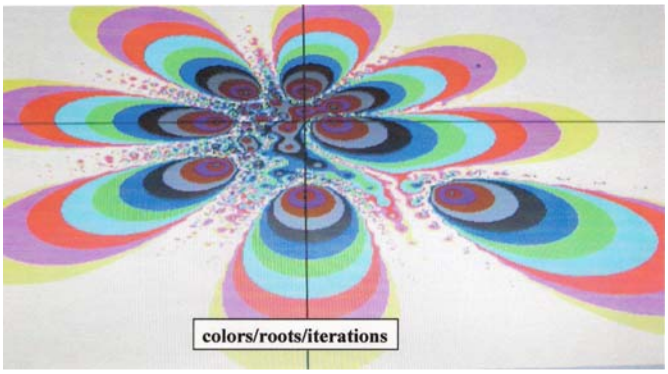
```
function modul;
begin
modul:=sqrt(sqr(a.re)+sqr(a.im));
end;

function permut;
var i,p:integer;
begin
    if k=0 then permut:=1
    else
    begin
    p := 1;
    for i:=n-k+1 to n do
    p := p * i;
    permut:=p;
     end;
end;

procedure Derivk;
Var p,q:complex;
    j:integer;
begin
    inminco(permut(n,k),a[n],p);
    for j:=1 to n-k do
       begin
         mult(p,z,p);
         inminco(permut(n-j,k),a[n-j],q);
         add(p,q,p);
        end;
        derk:=p;
end;
end.
```

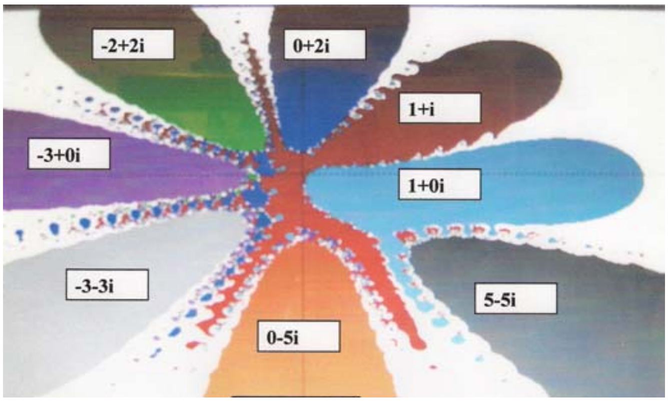
4. Examples

The following figures are generated by the Pascal program and presents an eight degree polynomial case and a four degree polynomial basins of attraction and the roots.

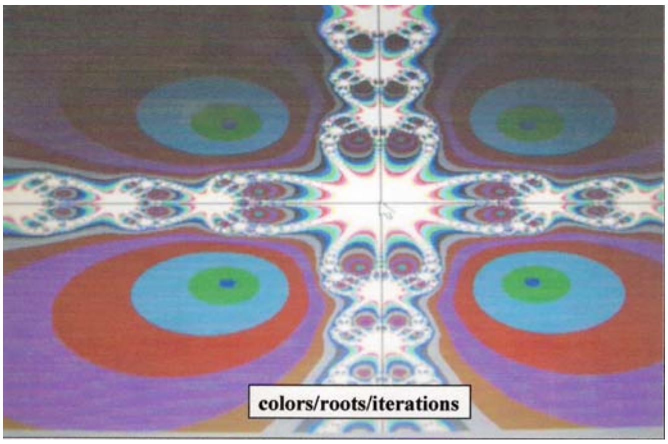
$$P(Z) = Z^8 + (1+8i)Z^7 + (-22+27i)Z^6 + (-105+70i)Z^5 + (-271+185i)Z^4 +$$

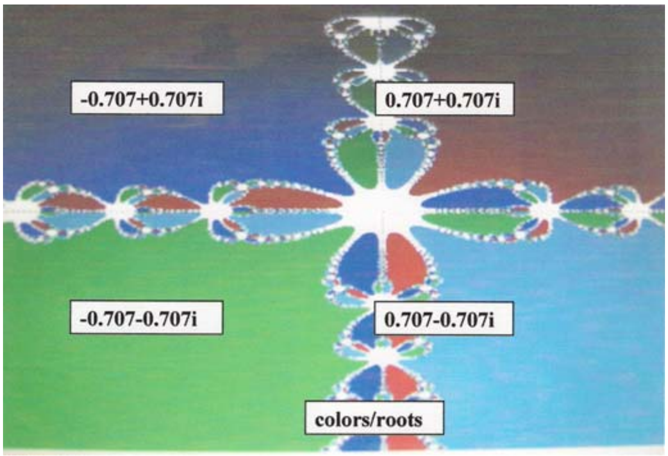
$$(-346+872i)Z^3 + (1282+1658i)Z^2 + (3060-2820i)Z - 3600$$





P(Z)= $Z^4 + 1$





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