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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} *anniversary*

Integral mean for fuzzy random variables

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ABSTRACT. In this paper we give some convergence properties with respect to weakly convergence for integral mean of fuzzy random variables.

1. INTRODUCTION AND PRELIMINARIES

Let X be a real separable reflexive Banach space with dual X^* and $\langle\cdot,\cdot\rangle$ the dual operations between X and X^*. Let 2^X denote the family of all nonempty subsets of X and $\sigma(A, x^*) = \sup\{\langle x^*, x\rangle ; x \in A\}$ the support function of $A \subset X$. Also, we denote by $\mathcal{P}_f(X)$ the family of all nonempty closed subsets of X, by $\mathcal{P}_{bfc}(X)$ the family of all nonempty bounded closed convex subsets of X, and by $\mathcal{P}_{wkc}(X)$ the family of all nonempty weakly compact convex subsets of X. For $A, B \in \mathcal{P}_f(X)$, let $H(A, B)$ denote the Hausdorff metric of A and B defined by

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

where $d(a,B) = \inf_{b\in B}\|a-b\|$ and $\|\cdot\|$ is the norm of X. If A, B are convex sets, then

$$H(A,B) = \sup_{\|x^*\|\le 1} |\sigma(A,x^*) - \sigma(B,x^*)|$$

Let (T, Σ, μ) be a complete finite measurable space and $F : T \to \mathcal{P}_f(X)$ a set valued function. F is said to be measurable if

$$F^{-1}(A) = \{t \in T; F(t) \cap A \neq \varnothing\} \in \Sigma$$

for every $A \in \mathcal{P}_f(X)$. A function $f : T \to X$ is called an integrable selector of F if f is Bochner integrable and $f(t) \in F(t), t \in T$. Let

$$S_F = \{f; f : T \to X \text{ is an integrable sector}\}.$$

For $A \in \Sigma$, the Aumann-Bochner integral of F is defined by

$$\int_A F(t)d\mu(t) = \left\{\int_A f(t)d\mu(t); f \in S_F\right\},$$

where $\int_A f(t)d\mu(t)$ is the Bochner integral of f.

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F is said to be integrable if $S_F \neq \varnothing$. F is said to be integrable bounded if $|F(t)| = \sup_{x \in F(t)} \|x\|$ is integrable.

We know (see [6]) that $F : T \to \mathcal{P}_{wkc}(X)$ is measurable if and only if for every $x^* \in X^*$, $t \to \sigma(F(t), x^*)$ is measurable. Moreover, if F is measurable and integrable bounded then $\int_A F(t)d\mu(t) \in \mathcal{P}_{wkc}(X)$ and $\sigma(\int_A F(t)d\mu(t), x^*) = \int_A \sigma(F(t), x^*)d\mu(t)$ for every $x^* \in X^*$.

The definition of Aumann-Bochner integral was introduced by Aumann in [1] on $\mathbb{R}^n$ and was generalized by Hiai [5] and Papageorgiou [8] on Banach spaces.

In this paper a fuzzy vector $u \in \mathcal{F}_{wkc}(X)$ is a function $u : X \to [0,1]$ for which the α- level set $[u]^\alpha$ of u, defined by $[u]^\alpha = \{x \in X; u(x) \geq \alpha\}$ is nonempty, weakly compact convex subset of X for all $\alpha \in [0,1]$. Also $[u]^0 = \overline{\{x \in X; u(x) > 0\}}$ is weakly compact.

For two fuzzy vectors $u, v \in \mathcal{F}_{wkc}(X)$ we can define a distance $D : \mathcal{F}_{wkc}(X) \times \mathcal{F}_{wkc}(X) \to \mathbb{R}_+$ by

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} H([u]^\alpha, [v]^\alpha).$$

A function $u : X \to [0,1]$ is said to be a Lipschitz fuzzy vectorr if it satisfies the following:

(1) $[u]^\alpha$ is bounded closed convex;
(2) there exists a constant $L > 0$ such that

$$H([u]^\alpha, [u]^\beta) \leq L\,|\alpha - \beta| \text{ for all } \alpha, \beta \in [0,1]$$

If we put $\mathcal{F}_L(X) = \{u; u \text{ is Lipschitz fuzzy vector}\}$ then we have that $\mathcal{F}_L(X) \subset \mathcal{F}_{wkc}(X)$. $F : T \to \mathcal{F}_{wkc}(X)$ is said to be a measurable fuzzy mapping (or a fuzzy random variable) if $[F]^\alpha(t) = \{x \in X; F(t) \geq \alpha\}$ is a measurable set-valued mapping for every $\alpha \in [0,1]$.

F is said to be integrable if for every $A \in \Sigma$ there exists a $u_A \in \mathcal{F}_{wkc}(X)$ such that $[u_A]^\alpha = \int_A [F]^\alpha(t)d\mu(t)$ for all $\alpha \in [0,1]$. We call $u_A = \int_A F(t)d\mu(t)$ the integral of F on A. Therefore, $\int_A F(t)d\mu(t) = u_A$ if and only if $[u_A]^\alpha = [\int_A F(t)d\mu(t)]^\alpha = \int_A [F]^\alpha(t)d\mu(t)$.

A sequence $\{u_n\} \subset \mathcal{F}_{wkc}(X)$ is said to be *weakly convergent* to $u \in \mathcal{F}_{wkc}(X)$ if

$$\sigma([u_n]^\alpha, x^*) \to \sigma([u]^\alpha, x^*) \text{ as } n \to \infty$$

for all $\alpha \in [0,1]$ and $x^* \in X^*$. We denote by $u_n \xrightarrow{w} u$ the weakly convergence.

In this paper we give some convergence properties with respect to weakly convergence for integral mean of fuzzy random variables. The properties of integral mean for fuzzy random variables with respect to the metric D was studied in [2]. The set-valued versions of the results in [2] were given in [10].

2. The convergence of integral mean

Let $F : T \to \mathcal{F}_{wkc}(X)$ be an integrable bounded fuzzy random variable. Then the fuzzy mapping $M_F : T \to \mathcal{F}_{wkc}(X)$ given by

$$M_F(A) = \frac{1}{\mu(A)} \int_A F(t)d\mu(t),$$

where $A \in \Sigma$ and $\mu(A) \neq 0$ is called the fuzzy integral mean of F.

Remark 2.1. For every $\alpha \in [0,1]$ and $A \in \Sigma$ with $\mu(A) \neq 0$, we have that

$$[M_F(A)]^\alpha = \frac{1}{\mu(A)} \int_A [F]^\alpha(t) d\mu(t).$$

Indeed we have

$$\begin{aligned} [M_F(A)]^\alpha &= \left[\frac{1}{\mu(A)} \int_A F(t) d\mu(t)\right]^\alpha = \frac{1}{\mu(A)} \left[\int_A F(t) d\mu(t)\right]^\alpha \\ &= \frac{1}{\mu(A)} \int_A [F]^\alpha(t) d\mu(t). \end{aligned}$$

Proposition 2.1. *Let $F : T \to \mathcal{F}_L(X)$ be a fuzzy random variable such that satisfy the following:*

(1) *there exists an integrable function $g : T \to \mathbb{R}_+$ such that*
$$|[F]^0(t)| \leq g(t),$$
(2) *there exists an integrable function $h : T \to \mathbb{R}_+$ such that*
$$H([F]^\alpha(t), [F]^\beta(t)) \leq h(t) |\alpha - \beta| \text{ for all } \alpha, \beta \in [0,1].$$

Then, for each $A \in \Sigma$ with $\mu(A) \neq 0$, there exists $u_A \in F_L(X)$ such that

$$[u_A]^\alpha = \left[\frac{1}{\mu(A)} \int_A F(t) d\mu(t)\right]^\alpha = \frac{1}{\mu(A)} \int_A [F]^\alpha(t) d\mu(t)$$

i.e. $M_F(A) \in F_L(X)$.

Proof. According to theorem 4.5 in [11], then exists a $u_A : X \to [0,1]$ such that $[u_A]^\alpha \in \mathcal{P}_{bfc}(X)$ and satisfies

$$[u_A]^\alpha = \left[\int_A F(t) d\mu(t)\right]^\alpha = \int_A [F]^\alpha(t) d\mu(t)$$

Hence

$$\begin{aligned} H([M_F(A)]^\alpha, [M_F(A)]^\beta) &= H\left(\left[\frac{1}{\mu(A)} \int_A F(t) d\mu(t)\right]^\alpha, \left[\frac{1}{\mu(A)} \int_A F(t) d\mu(t)\right]^\beta\right) \\ &= H\left(\frac{1}{\mu(A)} \left[\int_A F(t) d\mu(t)\right]^\alpha, \frac{1}{\mu(A)} \left[\int_A F(t) d\mu(t)\right]^\beta\right) \\ &= \frac{1}{\mu(A)} H\left(\int_A [F]^\alpha(t) d\mu(t), \int_A [F]^\beta(t) d\mu(t)\right) \\ &\leq \frac{1}{\mu(A)} \int_A H\left([F]^\alpha(t), [F]^\beta(t)\right) d\mu(t) \\ &\leq \frac{1}{\mu(A)} \left(\int_A h(t) d\mu(t)\right) |\alpha - \beta| \end{aligned}$$

we have $M_F(A) \in \mathcal{F}_L(X)$ and the proof is complete. □

Theorem 2.1. *Let $F : T \to \mathcal{F}_L(X)$ be a fuzzy random variable such that satisfy the following:*

(1) *there exists an integrable function* $g : T \to \mathbb{R}_+$ *such that*

$$|[F]^0(t)| \leq g(t),$$

(2) *there exists an integrable function* $h : T \to \mathbb{R}_+$ *such that*

$$H([F]^\alpha(t), [F]^\beta(t)) \leq h(t)\,|\alpha - \beta| \text{ for all } \alpha, \beta \in [0,1]\,.$$

Then the level-application $\alpha \to [M_F(A)]^\alpha$ *is H-continuous on* $[0,1]$.

Proof. Let $A \in \Sigma$ *with* $\mu(A) \neq 0$ and $\alpha_n, \alpha \in [0,1]$ such that $\alpha_n \to \alpha$ as $n \to \infty$. Since

$$\begin{aligned}
& H([M_F(A)]^{\alpha_n}, [M_F(A)]^\alpha) \\
= \;& H\left(\left[\frac{1}{\mu(A)}\int_A F(t)d\mu(t)\right]^{\alpha_n}, \left[\frac{1}{\mu(A)}\int_A F(t)d\mu(t)\right]^{\alpha}\right) \\
= \;& H\left(\frac{1}{\mu(A)}\left[\int_A F(t)d\mu(t)\right]^{\alpha_n}, \frac{1}{\mu(A)}\left[\int_A F(t)d\mu(t)\right]^{\alpha}\right) \\
= \;& \frac{1}{\mu(A)} H\left(\int_A [F]^{\alpha_n}(t)d\mu(t), \int_A [F]^{\alpha}(t)d\mu(t)\right) \\
\leq \;& \frac{1}{\mu(A)}\int_A H\left([F]^{\alpha_n}(t), [F]^{\alpha}(t)\right) d\mu(t) \\
\leq \;& \frac{1}{\mu(A)}\left(\int_A h(t)d\mu(t)\right)|\alpha_n - \alpha|
\end{aligned}$$

we infer that $H([M_F(A)]^{\alpha_n}, [M_F(A)]^\alpha) \to 0$ as $n \to \infty$, and so $\alpha \to [M_F(A)]^\alpha$ *is H-continuous on* $[0,1]$. □

Theorem 2.2. *Let* $F_n : T \to \mathcal{F}_{wkc}(X)$, $n \in \mathbb{N}$, *be a sequence of integrably bounded fuzzy random variables such that* $F_n \xrightarrow{w} F \in \mathcal{F}_{wkc}(X)$ *as* $n \to \infty$. *Then* $M_{F_n}(A) \xrightarrow{w} M_F(A)$ *for all* $A \in \Sigma$ *with* $\mu(A) \neq 0$.

Proof. Since $F_n \xrightarrow{w} F$ as $n \to \infty$ then, for every $\varepsilon > 0$ and $x^* \in X^*$ there exists $n_0 \in \mathbb{N}$ such that

$$|\sigma([F_n]^\alpha(t), x^*) - \sigma([F]^\alpha(t), x^*)| < \varepsilon$$

for all $n \geq n_0$. Further, if $A \in \Sigma$ with $\mu(A) \neq 0$ then, for $n \geq n_0$, we have

$$\begin{aligned}
& |\sigma([M_{F_n}(A)]^\alpha, x^*) - \sigma([M_F(A)]^\alpha, x^*)| \\
= \;& \frac{1}{\mu(A)}\left|\sigma\left(\left[\int_A F_n(t)d\mu(t)\right]^\alpha, x^*\right) - \sigma\left(\left[\int_A F(t)d\mu(t)\right]^\alpha, x^*\right)\right| \\
= \;& \frac{1}{\mu(A)}\left|\int_A \sigma([F_n]^\alpha(t), x^*)\,d\mu(t) - \int_A \sigma([F]^\alpha(t), x^*)\,d\mu(t)\right| \\
= \;& \frac{1}{\mu(A)}\left|\int_A [\sigma([F_n]^\alpha(t), x^*) - \sigma([F]^\alpha(t), x^*)]d\mu(t)\right| \\
\leq \;& \frac{1}{\mu(A)}\int_A |\sigma([F_n]^\alpha(t), x^*) - \sigma([F]^\alpha(t), x^*)|\,d\mu(t) < \varepsilon.
\end{aligned}$$

Therefore $\lim_{n\to\infty} \sigma([M_{F_n}(A)]^\alpha, x^*) = \sigma([M_F(A)]^\alpha, x^*)$ and hence $M_{F_n}(A) \xrightarrow{w} M_F(A)$ as $n \to \infty$. □

REFERENCES

[1] Aumann, R.J., *Integrals of set-valued functions*, J. Math. Anal. Appl. 12(1965), 1-11

[2] Chalco-Cano, Y., Rojas-Medar, M.A. and Román-Flores, H., *M-convex fuzzy mappings and fuzzy integral mean*, Comput. Math. Appl. **40** (2000), 1117-1126

[3] Gasiński, L. and Papageorgiou, N. S., *Nonlinear Analysis*, Taylor & Francis, New York, 2005

[4] Gil, M., López-Diaz, M. and Ralescu, D.A., *Overview on the development of fuzzy random variables*, Fuzzy Sets and Systems **157** (2006), 2546-2557

[5] Hiai, F. and Umegaki, H., *Integrals, conditional expectations and martingales of multivalued functions*, J. Multivar. Anal. **7** (1977), 149-182

[6] Hu, S. and Papageorgiou, N. S., *Handbook of multivalued analysis*, Volume I: Theory, Kluwer Academic Publisher, 1997

[7] Kim, B.K. and Kim, J.H., *Stochastic integrals of set-valued processes and fuzzy processes*, J. Math. Anal. Appl. **236** (1999), 480-502

[8] Papageorgiou, N. S., *On the theory of Banach space valued multifunctions*, J. Multivar. Anal., **17** (1985), 185-206

[9] Puri, M.L. and Ralescu, D.A., *Fuzzy random variables*, J. Math. Anal. Appl. **114** (1987), 301-317

[10] Ślepak, W., *On multivalued integral mean*, Demonstratio Mathematica **31** (1998), 339-344

[11] Xue, X., Minghu, H. and Ming, M., *Randon fuzzy number integrals in Banach spaces*, Fuzzy Sets and Systems **66** (1994), 97-111

[12] Xue, X., Wang, X. and Wu, L., *On the convergence and representation of random fuzzy number integrals*, Fuzzy Sets and Systems **103** (1999), 115-125

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