CREATIVE MATH. & INF. **17** (2008), No. 3, 313 - 318
Online version at http://creative-mathematics.ubm.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} anniversary

Some bivariate piecewise operators

MARIUS BIROU

ABSTRACT. In this paper we construct some bivariate interpolation operators using chains of bivariate piecewise Hermite projectors which are parametrical extensions of corresponding univariate projectors. We study the convergence of these operators.

1. INTRODUCTION

Boolean methods in multivariate approximation were introduced by Gordon W.J. in 1969 in [6]. These results were extended by Delvos F.J., Posdorf H., Schempp W. (see [5]).

Let X, Y be the linear spaces on $\mathbb{R}$.

The linear operator P defined on space X is called projector if $P^2 = P$.

The operator $P^C = I - P$, where I is identity operator, is called the remainder projector of P.

The set of interpolation points of projector P is denoted by $\mathcal{P}(P)$.

Proposition 1.1. *If P, Q are commutative projectors then we have*

$$\mathcal{P}(P \oplus Q) = \mathcal{P}(P) \cup \mathcal{P}(Q) \tag{1.1}$$

If P_1, P_2 are projectors on space X, we define relation "$\leq$":

$$P_1 \leq P_2 \Leftrightarrow P_1P_2 = P_1 \tag{1.2}$$

Proposition 1.2. [5] *Let $r \in \mathbb{N}$, $P_1, \ldots, P_r$ projectors on $C(X)$ and $Q_1, \ldots, Q_r$ projectors on $C(Y)$. Let $P_1', \ldots, P_r'$, $Q_1'', \ldots, Q_r''$ be the corresponding parametric extensions. We assume that*

$$P_1 \leq P_2 \leq \cdots \leq P_r, \quad Q_1 \leq Q_2 \leq \cdots \leq Q_r \tag{1.3}$$

We have that

$$B_r = P_1'Q_r'' \oplus P_2'Q_{r-1}'' \oplus \cdots \oplus P_r'Q_1'' \tag{1.4}$$

is projector and it has representation

$$B_r = \sum_{m=1}^{r} P_m'Q_{r+1-m}'' - \sum_{m=1}^{r-1} P_m'Q_{r-m}'' \tag{1.5}$$

Moreover, we have

$$B_r^C = P_r'^C + P_{r-1}'^C Q_1''^C + \cdots + P_1'^C Q_{r-1}''^C + Q_r''^C - (P_r'^C Q_1''^C + \cdots + P_1'^C Q_r''^C) \tag{1.6}$$

Received: 27.10.2008. In revised form: 11.05.2009. Accepted: 22.05.2009.
2000 *Mathematics Subject Classification.* 41A05, 41A63.
Key words and phrases. *Bivariate interpolation, piecewise Hermite interpolation, order of convergence.*
This work has been supported by MEC under CNCSIS grant, type A, code 1470/2007-2008.

where $P^C = I - P$, I is identity operator.

If the projectors P_i and Q_j are univariate Lagrange interpolation projectors we obtain bivariate Biermann interpolation projector (see [5]). The operator B_r was studied by the author in the case when P_i and Q_j are univariate Hermite and Birkhoff interpolation projectors ([3]), [4]). The bivariate polynomial interpolation schemes generated by operator B_r are used as local approximation tools. To obtain global bivariate interpolation schemes we can use piecewise interpolation projectors instead of polynomial projectors of Lagrange, Hermite or Birkhoff type. Baszenski and Delvos used periodic spline interpolation projectors to construct bivariate global approximation method (see [5]). In this article we use piecewise Hermite interpolation projectors to obtain bivariate interpolation schemes which are global. These interpolation schemes have applications in finite element method.

In section 2 we give some preliminaries notions and results related to univariate piecewise Hermite interpolation projectors from [1], [2].

In section 3 we construct the bivariate piecewise Hermite interpolation B_r and study the interpolation properties and the order of convergence.

2. Preliminaries

Let $T \in \mathbb{R}, T > 0$ and $\Delta :\ 0 = x_0 < x_1 < \cdots < x_{N+1} = T$ a uniform partition of interval $[0, T]$ with $h = T/(N+1)$.

Definition 2.1. [1] If r and p are positive integer, $1 \leq p \leq \infty$ then $PC^{r,p}[0, T]$ is defined as being the set of the functions which satisfy the following conditions:

i) f is $(r-1)$ times continuous and differentiable on $[0, T]$;
ii) there is s_i, $0 \leq i \leq L+1$ with $0 = s_0 < s_1 < \cdots < s_{L+1} = T$ so that on each open subinterval (s_i, s_{i+1}), $0 \leq i \leq L$, $f^{(r-1)}$ is continuous and differentiable;
iii) the norm L_p of the derivative $f^{(r)}$ is finite, i.e.

$$\|D^r f\|_p = \left(\sum_{i=0}^{L} \int_{s_i}^{s_{i+1}} |(D^r f)(x)|^p dx \right)^{1/p} < \infty.$$

In the case of $p = \infty$ we have

$$\|D^r f\|_\infty = \max_{0 \leq i \leq L} \sup_{x \in (s_i, s_{i+1})} |(D^r f)(x)| < \infty.$$

For Δ fixed, we define the set $H_m(\Delta) = \{h \in C^{m-1}[0, T] |\ h$ is a polynomial of degree at most $(2m-1)$ on each subinterval $[x_i, x_{i+1}]$, $0 \leq i \leq N\}$. We notice that the space $H_m(\Delta)$ has the dimension $m(N+2)$.

Definition 2.2. [1] If $f \in C^{m-1}[0, T]$ then we say that $H_m^\Delta f$ is the Hermite interpolation function of f respect to the set $H_m(\Delta)$ if $H_m^\Delta f \in H_m(\Delta)$ with $D^k(H_m^\Delta f)(x_i) = (D^k f)(x_i) = x_i^{(k)}$, $0 \leq i \leq N+1$, $0 \leq k \leq m-1$.

If $f \in C^{m-1}[0, T]$ then the function $H_m^\Delta f$ exists unique and has representation

$$(H_m^\Delta f)(x) = \sum_{i=0}^{N+1} \sum_{j=0}^{m-1} h_{m,i,j}(x) f^{(j)}(x_i) \tag{2.7}$$

where $h_{m,i,j}$, $0 \le i \le N+1$, $0 \le j \le m-1$ are cardinal functions from $H_m(\Delta)$ and which satisfy the conditions

$$h_{m,i,j}^{(\nu)}(x_\mu) = \delta_{i\mu}\delta_{j\nu}, \quad 0 \le \nu \le m-1,\ 0 \le \mu \le N+1.$$

We have an integral representation for the remainder term of piecewise Hermite interpolation (see [2])

$$f(x) - (H_m^{\Delta} f)(x) = \int_0^T G_N(x,s) f^{(2m)}(s) ds$$

with the Peano kernel function

$$G_N(x,s) = (H_m^{\Delta})^c \left[\frac{(x-s)_+^{2m-1}}{(2m-1)!} \right]$$

where $z_+ = z$ if $z \ge 0$ and $z_+ = 0$ if $z < 0$. The application of $(H_m^{\Delta})^c$ to $\frac{(x-s)_+^{2m-1}}{(2m-1)!}$ is considered respect to the variable x.

Theorem 2.1. [1] *If* $f \in PC^{2m,\infty}[0,T]$ *then*

$$\|f - H_m^{\Delta} f\|_\infty = O(h^{2m-\frac{1}{2}}), \quad h \to 0.$$

It follows there exists a positive constant c such that

$$\sup \left\{ \int_0^T |G_N(x,s)|\, ds, x \in [0,T] \right\} = ch^{2m-\frac{1}{2}}$$

3. Main results

Let $N \in \mathbb{N}$, $N > 1$ and $D = [0,T]$, $T > 0$. Let $\Delta^k :\ 0 = x_0 < x_1 < \cdots < x_{N^k} = T$ a uniform partition of the interval with the step T/N^k. The univariate piecewise Hermite interpolation projectors

$$H_{m,N^k} := H_m^{\Delta_k}, \quad k \in \mathbb{N}$$

are commuting in $PC^{2m,\infty}[0,T]$ and verify the order relation

$$H_{m,N^k} \le H_{m,N^l}, \quad k \le l. \tag{3.8}$$

Definition 3.3. [1] The space $PC^{p,r}(D^2)$ is defined as being the space of the bivariate functions which satisfy the following conditions

i) f is $(r-1)$ times continuous differentiable on D^2, i.e. there exist the derivatives $D_x^\mu D_y^\nu$, $0 \le \mu + \nu \le r-1$ and they are continuous on D^2;
ii) there are s_i, $0 \le i \le L+1$, and v_j, $0 \le j \le R+1$ with $0 = s_0 < s_1 < \ldots < s_{L+1} = T$ and $0 = v_0 < v_1 < \ldots < v_{R+1} = T$ so that on the open rectangular domain $(s_i, s_{i+1}) \times (v_j, v_{j+1})$, $0 \le i \le L$, $0 \le j \le R$ and for every $0 \le \mu, \nu \le r-1$ with $\mu + \nu = r-1$, the derivatives $D_x^\mu D_y^\nu$ are continuous and differentiable;
iii) for every $0 \le \mu, \nu \le r-1$ with $\mu + \nu = r-1$ the norm L_p of the derivative $D_x^\mu D_y^\nu$ is finite

$$\left\| D_x^\mu D_y^\nu \right\|_p = (\sum_{i=0}^{L} \sum_{j=0}^{R} \int_{s_i}^{s_{i+1}} \int_{v_j}^{v_{j+1}} \left| (D_x^\mu D_y^\nu f)(x,y) \right|^p dxdy)^{1/p} < \infty.$$

For the particular case $p = \infty$ it reduces to

$$\left\|D_x^\mu D_y^\nu f\right\|_\infty = \max_{\substack{0\le i\le L\\0\le j\le R}} \sup_{(x,y)\in(s_i,s_{i+1})\times(v_j,v_{j+1})} \left|D_x^\mu D_y^\nu f(x,y)\right| < \infty$$

The parametrical extensions

$$(H'_{m,N^k} f)(x,y) = \sum_{i=0}^{N^k+1}\sum_{p=0}^{m-1} h_{m,i,p}(x) f^{(p,0)}(x_i,y)$$

$$(H''_{m,N^l} f)(x,y) = \sum_{j=0}^{N^l+1}\sum_{q=0}^{m-1} h_{m,j,q}(y) f^{(0,q)}(x,y_j)$$

generate a distributive lattice of interpolation projectors on $PC^{2m,\infty}(D^2)$ and a boolean algebra of projectors which contain the remainder projectors.

The tensor product interpolation operator is given by

$$(H'_{m,N^k} H''_{m,N^l} f)(x,y) = \sum_{i=0}^{N^k+1}\sum_{j=0}^{N^l+1}\sum_{p=0}^{m-1}\sum_{q=0}^{m-1} h_{m,i,p}(x) h_{m,j,q}(y) f^{(p,q)}(x_i,y_j)$$

We construct the piecewise Hermite blending interpolation projector of order r

$$B_r^N = H'_{m,N} H''_{m,N^r} \oplus H'_{m,N^2} H''_{m,N^{r-1}} \oplus \cdots \oplus H'_{m,N^r} H''_{m,N}. \tag{3.9}$$

Proposition 3.3. *The interpolation projector B_r^N has the following interpolation properties*

$$(B_r^N f)^{(p,q)}\left(i\frac{T}{N^n}, j\frac{T}{N^{r+1-n}}\right) = f^{(p,q)}\left(i\frac{T}{N^n}, j\frac{T}{N^{r+1-n}}\right), \tag{3.10}$$

$$0 \le i \le N^n,\ 0 \le j \le N^{r+1-n},\ 1 \le n \le r,\ 0 \le p,q \le m-1.$$

Proof. Taking into account

$$(H'_{m,N^k} H''_{m,N^l} f)^{(p,q)}\left(i\frac{T}{N^k}, j\frac{T}{N^l}\right) = f^{(p,q)}\left(i\frac{T}{N^k}, j\frac{T}{N^l}\right)$$

$$0 \le i \le N^k,\ 0 \le j \le N^l,\ 0 \le p,q \le m-1$$

and

$$\mathcal{P}(B_r^N) = \bigcup_{n=1}^{r} \mathcal{P}(H'_{m,N^n} H''_{m,N^{r+1-n}})$$

we obtain the relation (3.10). □

Remark 3.1. If $r = 1$ then $B_1^N = H'_{m,N} H''_{m,N}$ is piecewise Hermite tensor product interpolation projector.

If $r = 2$ then

$$B_2^N = H'_{m,N} H''_{m,N^2} + H'_{m,N^2} H''_{m,N} - H'_{m,N} H''_{m,N}$$

is named the discrete blending piecewise Hermite interpolation projector.

Theorem 3.2. *If $f \in C^{2m,\infty}(D^2)$ then*

$$\|f - B_2^N f\|_\infty = O\left(\left(\frac{1}{N^2}\right)^{2m-\frac{1}{2}}\right), \quad N \to \infty.$$

Proof. From the formula (1.6) we have

$$(B_2^N)^c = (H'_{m,N^2})^c + (H''_{m,N^2})^c + (H'_{m,N})^c (H''_{m,N})^c$$
$$-(H'_{m,N^2})^c (H''_{m,N})^c - (H'_{m,N})^c (H'_{m,N^2})^c.$$

It follows

$$|f(x,y) - (B_2^N f)(x,y)|$$
$$\le \int_0^T |G_{N^2}(x,s)| ds \|f^{(2m,0)}\|_\infty + \int_0^T |G_{N^2}(y,t)| dt \|f^{(0,2m)}\|_\infty$$
$$+ \int_0^T \int_0^T |G_N(x,s)||G_N(y,t)| ds dt \|f^{(2m,2m)}\|_\infty$$
$$+ \int_0^T \int_0^T |G_{N^2}(x,s)||G_N(y,t)| ds dt \|f^{(2m,2m)}\|_\infty$$
$$+ \int_0^T \int_0^T |G_N(x,s)||G_{N^2}(y,t)| ds dt \|f^{(2m,2m)}\|_\infty$$
$$\le c\left(\frac{T}{N^2}\right)^{2m-\frac{1}{2}} \|f^{(2m,0)}\|_\infty + c\left(\frac{T}{N^2}\right)^{2m-\frac{1}{2}} \|f^{(0,2m)}\|_\infty$$
$$+c^2\left(\frac{T}{N}\right)^{2m-\frac{1}{2}}\left(\frac{T}{N}\right)^{2m-\frac{1}{2}} \|f^{(2m,2m)}\|_\infty + c^2\left(\frac{T}{N^2}\right)^{2m-\frac{1}{2}}\left(\frac{T}{N}\right)^{2m-\frac{1}{2}} \|f^{(2m,2m)}\|_\infty$$
$$+c^2\left(\frac{T}{N}\right)^{2m-\frac{1}{2}}\left(\frac{T}{N^2}\right)^{2m-\frac{1}{2}} \|f^{(2m,2m)}\|_\infty$$
$$= O\left(\left(\frac{1}{N^2}\right)^{2m-\frac{1}{2}}\right), \quad N \to \infty.$$

□

Theorem 3.3. *If $f \in C^{2m,\infty}(D^2)$ then*

$$\|f - B_r^2 f\|_\infty = O\left(r\left(\frac{1}{2^r}\right)^{2m-\frac{1}{2}}\right), \quad r \to \infty.$$

Proof. From the formula (1.6) we have

$$(B_r^2)^c = (H'_{m,2^r})^c + (H''_{m,2^r})^c + \sum_{n=1}^{r-1} (H'_{m,2^{r-n}})^c (H''_{m,2^n})^c$$
$$-\sum_{n=1}^{r} (H'_{m,2^{r+1-n}})^c (H''_{m,2^n})^c.$$

It follows

$$|f(x,y) - (B_r^2 f)(x,y)|$$

$$
\begin{aligned}
&\leq \int_0^T |G_{2^r}(x,s)|ds|f^{(2m,0)}\|_\infty + \int_0^T |G_{2^r}(x,t)|dt\|f^{(0,2m)}\|_\infty \\
&\quad + \sum_{n=1}^{r-1} \int_0^T \int_0^T |G_{2^{r-n}}(x,s)||G_{2^n}(y,t)|dsdt\|f^{(2m,2m)}\|_\infty \\
&\quad + \sum_{n=1}^{r} \int_0^T \int_0^T |G_{2^{r+1-n}}(x,s)||G_{2^n}(y,t)|dsdt\|f^{(2m,2m)}\|_\infty \\
&\leq c\left(\frac{T}{2^r}\right)^{2m-\frac{1}{2}} \|f^{(2m,0)}\|_\infty + c\left(\frac{T}{2^r}\right)^{2m-\frac{1}{2}} \|f^{(0,2m)}\|_\infty \\
&\quad + \sum_{n=1}^{r-1} c\left(\frac{T}{2^{r-n}}\right)^{2m-\frac{1}{2}} c\left(\frac{T}{2^n}\right)^{2m-\frac{1}{2}} \|f^{(2m,2m)}\|_\infty \\
&\quad + \sum_{n=1}^{r} c\left(\frac{T}{2^{r+1-n}}\right)^{2m-\frac{1}{2}} c\left(\frac{T}{2^n}\right)^{2m-\frac{1}{2}} \|f^{(2m,2m)}\|_\infty \\
&= O\left(r\left(\frac{1}{2^r}\right)^{2m-\frac{1}{2}}\right), \quad r \to \infty.
\end{aligned}
$$

□

References

[1] Agarwal, R. P. and Wong, P. J. Y., *Error inequalities in polynomial interpolation and their applications*, Mathematics and its Applications 262, Kluwer Academic Publishers Group, Dordrecht, 1993

[2] Birkhoff, G., Schultz, M.H., Varga, R.S., *Piecewise Hermite interpolation in one and two variables with applications to partial differential equations*, Numerische Mathematik, **11** (1968), 232-256.

[3] Birou, M., *Biermann interpolation with Hermite information*, Studia Univ. Babeş Bolyai, **51** (2005), No. 3, 41- 55

[4] Birou, M., *Biermann interpolation of Birkhoff type*, Rev. Anal. Numer. Theor. Approx., **34** (2005), No. 1, 37-45

[5] Delvos, F. J. and Schemp, W., *Boolean methods in interpolation and approximation*, Pitman Research Notes in Math., Series 230 New York, 1989

[6] Gordon, W. S., *Distributive lattices and approximation of multivariate functions*, in Proc. Symp. Approximation with special emphasis on Spline Function, Madison, Wisc., 1969, (I.J. Schoenberg ed.), 223-277

Technical University of Cluj-Napoca
Department of Mathematics
Str Daicoviciu No 15
400020 Cluj-Napoca, România
E-mail address: Marius.Birou@math.utcluj.ro