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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} *anniversary*

On the stability of quartic type functional equation

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ABSTRACT. In this article we investigate the generalized Hyers-Ulam-Rassias stability for the quartic type functional equation $f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)+24f(y)-6f(x)$ by using the fixed point alternative and we shall obtain a better estimate for the difference in norm of a solution of equation and a sub-solution of equation.

1. INTRODUCTION

In 1940, S.M.Ulam [5] gave the following question concerning the stability of homomorfisms: *Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, there exists a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$.*

In the next year, D.H. Hyers [3] excellently answered the question of Ulam for the case where G_1 and G_2 are Banach space. Th. M. Rassias [11], T.Aoki [8], Z. Gajda [9] and Găvruţă [7] considered the stability problem with unbounded Cauchy differences. The stability phenomenon that was introduced and proved by Th. M. Rassias in [11] is called the generalized Hyers-Ulam-Rassias stability. These terminologies are also applied to the case of other functional equation.
In [14] V. Radu has the excellent idea to use the fixed point alternative to prove the generalized Hyers-Ulam-Rassias stability for functional equations.

Now, we consider the following functional equation:

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y) \qquad (1.1)$$

It easy to see that the function $f(x)=cx^4,\ c\in \mathbf{R}$ satisfies functional equation (1.1). Hence, it is natural that equation (1.1) is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function. The stability of equation (1.1) was obtained by S.H. Lee, S.M. Im and I.S. Hwang in [13].

Now we introduce another quartic type equation, that is,

$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)+24f(y) \qquad (1.2)$$

In [15] J. M. Rassias proved the Hyers-Ulam stability for the functional equation (1.2), using the direct method and proved that:

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Theorem 1.1. *Let X be a normed linear space and Y be a Banach space, on the real field. If a function $f : X \to Y$ satisfies the inequality*

$$\|f(x+2y) + f(x-2y) - 4f(x+y) - 4f(x-y) + 6f(x) - 24f(y)\| \leqslant \varepsilon$$

for all $x, y \in X$, with a constant $\varepsilon \geqslant 0$, then there exists a unique quartic mapping $c : X \to Y$ such that

$$\|f(x) - c(x)\| \leqslant \frac{17\varepsilon}{180} \tag{1.3}$$

for all $x \in X$. The function c is given by

$$c(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{4n}}$$

for all $x \in X$ and $n \in \mathbf{N}$.

In [16] Cădariu and Radu proved the generalized Hyers-Ulam-Rassias stability of functional equation (1.2) showing that

Theorem 1.2. *Let E be a (real or complex) normed space, F a Banach space. Consider $\varepsilon, \delta, p, q$ fixed numbers, such that $\varepsilon, \delta, p, q \geqslant 0$ and either $p, q < 4$ or $p, q > 4$. Suppose that the mapping $f : E \to F$ satisfies the inequality*

$$\|f(x+2y) + f(x-2y) - 4f(x+y) - 4f(x-y) + 6f(x) - 24f(y)\| \leqslant$$
$$\leqslant \delta(1-i) + \varepsilon(\|x\|^p + \|y\|^q) \text{ for all } x, y \in E,$$

where $i = 0$ for $p, q < 4$ and $i = 1$ for $p, q > 4$. Then there exist a unique quartic mapping $c : E \to F$ which satisfies the inequality

$$\|f(x) - c(x)\| \leqslant \frac{5\delta(1-i)}{6(2^4 - 2^q)} + \frac{\varepsilon}{24} \cdot \frac{2^4 + 2^q}{|2^4 - 2^q|} \cdot \|x\|^q, \ \forall x \in E$$

As a particular case of Theorem 1.2, for $\varepsilon = 0$ and $p = q = 0$ we obtain the result of Theorem 1.1 where the relation (1.3) become

$$\|f(x) - c(x)\| \leqslant \frac{\varepsilon}{18} \tag{1.4}$$

In this note we solve the equation (1.2) and prove the stability of functional equation (1.2) using the control function $\varphi(x, y)$ which satisfies the proper conditions, and as a particular case we obtain a better estimate for the difference in norm of a solution of equation and a sub-solution of equation.

2. A SOLUTION OF FUNCTIONAL EQUATION (1.2)

It is well known [1] that a function $f : X \to Y$ between real vector spaces is quadratic if and only if there exist a unique symmetric biadditive function B such that $f(x) = B(x, x)$ for all $x \in X$. The biadditive function B is given by

$$B(x, y) = \frac{1}{2}(f(x+y) - f(x-y)). \tag{2.5}$$

Throughout this section X and Y will be real vector spaces.

In [13] Lee proved the following Lemma

Lemma 2.1. *A function $f : X \to Y$ satisfies the functional equation (1.2) if and only if there exists a symmetric biquadratic function $F : X \times X \to Y$ such that $f(x) = F(x, x)$ for all $x \in X$.*

Lemma 2.2. *A function $f : X \to Y$ satisfies the functional equation (1.2) if and only if f satisfies the functional equation (1.1).*

Proof. ($\Rightarrow$) Substituting $x = y = 0$ in (1.2) yields $f(0) = 0$. Putting $x = 0$ in (1.2), we get

$$f(-y) = f(y). \tag{2.6}$$

Let us interchange x with y in (1.2) and using (2.6) we get

$$f(x+2y) + f(x-2y) + 6f(x) = 4f(x+y) + 4f(x-y) + 24f(y) \tag{2.7}$$

which is equation (1.1)

($\Leftarrow$) Substituting $x = y = 0$ in (1.1) yields $f(0) = 0$. Putting $x = 0$ in (1.1), we get

$$f(2y) + f(-2y) = 28f(y) + 4f(-y) \tag{2.8}$$

Replacing y with $-y$ in (2.8) we get

$$f(-y) = f(y) \tag{2.9}$$

Let us interchange x with y in (1.1) and using (2.9) we get that function f satisfies the functional equation (1.2) □

Using the previous lemmas we get the solution of equation (1.2), and that is:

Lemma 2.3. *A function $f : X \to Y$ satisfies the functional equation (1.2) if and only if there exist a symmetric biquadratic function $F : X \times X \to Y$ such that $f(x) = F(x, x)$ for all $x \in X$.*

3. Stability of equation (1.2)

For explicit later use, we state the following theorem:

Theorem 3.1. *(The alternative of fixed point) Suppose that we are given a complete generalized metric space (Ω, d) and $T : \Omega \to \Omega$ a L-contraction with $L \in [0, 1)$. Then, for each given $x \in \Omega$, either*

$$d\left(T^n x,\, T^{n+1} x\right) = \infty,\ \forall n \geqslant 0$$

or there exist a natural number n_0 such that

- *$d\left(T^n x,\, T^{n+1} x\right) < \infty$ for all $n \geqslant n_0$;*
- *The sequence $(T^n x)_{n \geqslant 0}$ is convergent to a fixed point $y*$ of T;*
- *$y*$ is the unique fixed point of T in the set $\Delta = \{y \in \Omega \,|\, d(T^{n_0} x, y) < \infty\}$;*
- *$d(y,\, y*) \leqslant \frac{1}{1-L} d(y,\, Ty)$ for all $y \in \Delta$.*

Utilizing the above-mentioned fixed point alternative, we now obtain our main result, i.e., the generalized Hyers-Ulam-Rassias stability of the functional equation (1.2).

From now on, let X be a real vector space and Y be a real Banach space. Given a mapping $f : X \to Y$, we set

$$Df(x, y) := f(x+2y) + f(x-2y) + 6f(x) - 4f(x+y) - 4f(x-y) - 24f(y)\,; \forall x, y \in X$$

Let $\varphi : X \times X \to [0, \infty)$ be a function such that

$$\lim_{n\to\infty} \frac{\varphi\left(\lambda_i^n x,\, \lambda_i^n y\right)}{\lambda_i^{4n}} = 0 \tag{3.10}$$

for all $x, y, z \in X$, where $\lambda_i = 2$ if $i = 0$ and $\lambda_i = \frac{1}{2}$ if $i = 1$, and

$$\varphi(0, -y) = \varphi(0, y), \ \forall x \in X \tag{3.11}$$

Theorem 3.2. *Suppose that a function $f : X \to Y$ satisfies the functional inequality*

$$\|Df(x, y)\| \leqslant \varphi(x, y) \tag{3.12}$$

for all $x, y \in X$ and $f(0) = 0$. If there exist $L = L(i) < 1$ such that the function

$$x \mapsto \psi(x) = \phi\left(0, \frac{x}{2}\right)$$

has the property

$$\psi(x) \leqslant L \cdot \lambda_i^4 \psi\left(\frac{x}{\lambda_i}\right) \tag{3.13}$$

for all $x \in X$, then there exists a unique quartic function $C : X \to Y$ such that the inequality

$$\|f(x) - C(x)\| \leqslant \frac{L^{1-i}(2 + 2L)}{3(1 - L)} \psi(x) \tag{3.14}$$

holds for all $x \in X$.

Proof. Consider the set

$$\Omega = \{g \mid g : X \to Y\}$$

and introduce the generalized metric on Ω,

$$d(g, h) = d_\psi(g, h) = \inf\{K \in (0, \infty) \mid \|g(x) - h(x)\| \leqslant K\psi(x), \ \forall x \in X\}.$$

It is easy to see that (Ω, d) is complete. Now we define a mapping $T : \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{\lambda_i^4} g(\lambda_i x), \quad \forall x \in X.$$

Note that for all $g, h \in \Omega$,

$$\begin{aligned} d(g, h) < K &\Rightarrow \|g(x) - h(x)\| \leqslant K\psi(x), \ \ x \in X \\ &\Rightarrow \left\| \frac{1}{\lambda_i^4} g(\lambda_i x) - \frac{1}{\lambda_i^4} h(\lambda_i x) \right\| \leqslant \frac{1}{\lambda_i^4} K\psi(\lambda_i x), \ x \in X \\ &\Rightarrow \left\| \frac{1}{\lambda_i^4} g(\lambda_i x) - \frac{1}{\lambda_i^4} h(\lambda_i x) \right\| \leqslant LK\psi(x), \ x \in X \\ &\Rightarrow d(Tg, Th) \leqslant LK. \end{aligned}$$

Hence we see that

$$d(Tg, Th) \leqslant Ld(g, h)$$

for all $g, h \in \Omega$, that is, T is a strictly contractive selfmapping of Ω with the Lipschitz constant L.

If we put $x = 0$ in (3.12) we get

$$\|f(2y) + f(-2y) - 28f(y) - 4f(-y)\| \leqslant \varphi(0, y) = \psi(2y) \tag{3.15}$$

If we substitute $y := -y$ in (3.15), we get,

$$\|f(2y) + f(-2y) - 28f(-y) - 4f(y)\| \leqslant \varphi(0, -y) = \psi(2y) \tag{3.16}$$

Then

$$24\left\|f(y)-f(-y)\right\| = \\ \left\|[f(2y)+f(-2y)-28f(y)-4f(-y)]-[f(2y)+f(-2y)-28f(-y)-4f(y)]\right\| \leqslant \\ \leqslant 2\psi(2y), \quad \forall y\in X,$$

and we obtain

$$\left\|f(y)-f(-y)\right\| \leqslant \frac{1}{12}\psi(2y), \quad \forall y\in X \tag{3.17}$$

Using (3.17), we get

$$2\left\|f(2y)-16f(y)\right\| = \\ \left\|(f(2y)+f(-2y)-4f(-y)-28f(y))+(f(2y)-f(-2y))+4(f(-y)-f(y))\right\| \leqslant \\ \leqslant \left\|f(2y)+f(-2y)-4f(-y)-28f(y)\right\|+\left\|f(2y)-f(-2y)\right\|+4\left\|f(y)-f(-y)\right\| \leqslant \\ \leqslant \psi(2y)+\tfrac{1}{12}\psi(4y)+\tfrac{1}{3}\psi(2y)=\tfrac{4+4L}{3}\psi(2y)$$

which yields:

$$\left\|f(2y)-16f(y)\right\| \leqslant \frac{2+2L}{3}\psi(2y) \leqslant \frac{2+2L}{3}\cdot 16L\psi(y) \tag{3.18}$$

which is reduced to

$$\left\|f(y)-\frac{1}{16}f(2y)\right\| \leqslant \frac{L(2+2L)}{3}\psi(y), \ \forall y\in X$$

that is, $d(f, Tf) \leqslant \frac{L(2+2L)}{3} < \infty$.

If we substitute $y := \frac{y}{2}$ in (3.18) and use (3.13), then we see that

$$\left\|f(y)-2^4 f\left(\frac{y}{2}\right)\right\| \leqslant \frac{2+2L}{3}\psi(y), \quad \forall y\in X$$

that is, $d(f, Tf) \leqslant \frac{2+2L}{3} < \infty$.

Now, from the fixed point alternative in both cases, it follows that there exists a fixed point C of T in Ω such that

$$C(x)=\lim_{n\to\infty}\frac{f(\lambda_i^n x)}{\lambda_i^{4n}}, \ \forall x\in X, \tag{3.19}$$

since $\lim\limits_{n\to\infty} d(T^n f, C)=0$.

To show that the function $C : X\to X$ is quartic, let us replace x and y by $\lambda_i^n x$ and $\lambda_i^n y$ in (3.10),respectively, and divide by λ_i^{4n}. Then it follows from (3.10) and (3.12) that

$$\left\|DC(x,y)\right\| = \lim_{n\to\infty}\frac{\left\|Df(\lambda_i^n x, \lambda_i^n y)\right\|}{\lambda_i^{4n}} \leqslant \lim_{n\to\infty}\frac{\phi(\lambda_i^n x, \lambda_i^n y)}{\lambda_i^{4n}}=0, \ \forall x,y\in X,$$

that is, C satisfies the functional equation (1.2). Therefore Lemma 2.2 guarantees that C is quartic.

According to the fixed point alternative, since C is the *unique* fixed point of Tin the set $\Delta=\{g\in\Omega:\ d(f, g)<\infty\}$, C is the unique function such that

$$\left\|f(x)-C(x)\right\| \leqslant K\psi(x)$$

for all $x \in X$ and some $K > 0$. Again using the fixed point alternative, we have

$$d(f, C) \leqslant \frac{1}{1-L} d(f, Tf)$$

and so we obtain the inequality

$$d(f, C) \leqslant \frac{L^{1-i}(2+2L)}{3(1-L)}$$

which yields the inequality (3.14). This completes the proof of theorem. □

From Theorem 3.2, we obtain the following corollary concerning the Hyers-Ulam stability of the functional equation (1.2).

Corollary 3.1. *Let X and Y be a normed space and a Banach space, respectively. Let $p \geqslant 0$ be given with $p \neq 4$. Assume that $\delta \geqslant 0$ and $\varepsilon \geqslant 0$ are fixed. Suppose that a function $f : X \to Y$ satisfies the functional inequality*

$$\|Df(x, y)\| \leqslant \delta + \varepsilon (\|x\|^p + \|y\|^p) \tag{3.20}$$

for all $x, y \in X$. Furthermore, assume that $f(0) = 0$ and $\delta = 0$ in (3.20) for the case $p > 4$. Then there exists a unique quartic function $C : X \to Y$ such that the inequality

$$\|f(x) - C(x)\| \leqslant \frac{2 + 2^{p-3}}{3(2^{4-p} - 1)}\delta + \frac{(2 + 2^{p-3})\varepsilon}{3(16 - 2^p)}\|x\|^p \tag{3.21}$$

holds for all $x \in X$, where $p < 4$, or the inequality

$$\|f(x) - C(x)\| \leqslant \frac{(2 + 2^{5-p})\varepsilon}{3(2^p - 16)}\|x\|^p \tag{3.22}$$

holds for all $x \in X$, where $p > 4$.

Proof. Let $\phi(x, y) := \delta + \varepsilon(\|x\|^p + \|y\|^p), \ \forall x, y \in X$. Then it follows that

$$\frac{\phi(\lambda_i^n x, \lambda_i^n y)}{\lambda_i^{4n}} = \frac{\delta}{\lambda_i^{4n}} + (\lambda_i^n)^{p-4}\varepsilon(\|x\|^p + \|y\|^p) \to 0$$

as $n \to \infty$, where $p < 4$, if $i = 0$ and $p > 4$, if $i = 1$, that is, the relation (3.12) is true.

Since the inequality

$$\frac{1}{\lambda_i^4}\psi(\lambda_i x) = \frac{\delta}{\lambda_i^4} + \frac{\lambda_i^{p-4}}{2^p}\varepsilon\|x\|^p \leqslant \lambda_i^{p-4}\psi(x)$$

holds for all $x \in X$, where $p < 4$ if $i = 0$ and $p > 4$ if $i = 1$, we see that the inequality (3.13) holds with either $L = 2^{p-4}$ or $L = 2^{4-p}$. Now the inequality (3.14) yields the inequality (3.21) and (3.22) which complete the proof of the corollary. □

The following corollary is the Hyers-Ulam stability of the functional equation (1.2).

Corollary 3.2. *Let X and Y be a normed space and a Banach space, respectively. Assume that $\theta \geqslant 0$ is fixed. Suppose that a function $f : X \to Y$ satisfies the functional inequality*

$$\|Df(x, y)\| \leqslant \theta \tag{3.23}$$

for all $x, y \in X$*. Then there exist a unique quartic function* $C : X \to Y$ *such that the inequality*

$$\|f(x) - C(x)\| \leqslant \frac{11}{720}\theta \tag{3.24}$$

holds for all $x \in X$*.*

Proof. In Corollary 3.1, putting $\delta := 0,\ p := 0$ and $\varepsilon := \frac{\theta}{2}$, we arrive at the conclusion of the corollary. □

Conclusion. We applied the alternative of fixed point to obtain the stability of equation (1.2), and is easy to see that our estimate (3.24) is better than estimate (1.3) and (1.4)

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