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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} anniversary

Fixed point theory for multivalued contractions on a set with two b-metrics

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ABSTRACT. The purpose of this paper is to present some fixed point results for multivalued contractions on a set with two b-metrics. The data dependence and the well-posedness of the fixed point problem are also discussed.

1. INTRODUCTION

The concept of b-metric space appeared in some works, such as N. Bourbaki, I.A. Bakhtin, S. Czerwik, etc. Several papers deal with the fixed point theory for singelvalued and multivalued operators in b-metric spaces (see [1], [2], [3], [8]). In the first part of the paper we will present a fixed point theorem for a multivalued contraction on b-metric space endowed with two b-metrics. Then, a strict fixed point result for multivalued contraction in b-metric spaces is proved. The last part contains several conditions under which the fixed point problem for a multivalued operator in a b-metric space is well-posed and a data dependence result is given.

2. NOTATIONS AND AUXILIARY RESULTS

The aim of this section is to present some notions and symbols used in the paper.

We will first give the definition of a b-metric space.

Definition 2.1. (Bakhtin [1], Czerwik [3]) Let X be a set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A pair (X, d) is called a b-metric space.

We give next some examples of b-metric spaces.

Example 2.1. (Berinde see [2])

The space $l_p(0 < p < 1)$,

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$$l_p = \{(x_n) \subset \mathbb{R} | \sum_{n=1}^{\infty} |x_n|^p < \infty\},$$

together with the function $d : l_p \times l_p \to \mathbb{R}$,

$$d(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{1/p},$$

where $x = (x_n), y = (y_n) \in l_p$ is a b-metric space.
By an elementary calculation we obtain: $d(x, z) \leq 2^{1/p}[d(x, y) + d(y, z)]$.
Hence $a = 2^{1/p} > 1$.

Example 2.2. (Berinde see [2])
The space $L_p(0 < p < 1)$ of all real functions $x(t)$, $t \in [0, 1]$ such that:

$$\int_0^1 |x(t)|^p dt < \infty,$$

is a b-metric space if we take

$$d(x, y) = (\int_0^1 |x(t) - y(t)|^p dt)^{1/p}, \text{ for each } x, y \in L_p,$$

The constant a is as in the previous example $2^{1/p}$.

We continue by presenting the notions of convergence, compactness, closedness and completeness in a b-metric space.

Definition 2.2. Let (X, d) be a b-metric space. Then a sequence $(x_n)_{n\in\mathbb{N}}$ in X is called:

(a): Cauchy if and only if for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for each $n, m \geq n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$.

(b): convergent if and only if there exists $x \in X$ such that for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ we have $d(x_n, x) < \varepsilon$. In this case we write $\lim_{n\to\infty} x_n = x$.

Remark 2.1. (1) The sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy if and only if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$, for all $p \in \mathbb{N}^*$.
(2) The sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to $x \in X$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.

Definition 2.3. (1) Let (X, d) be a b-metric space. Then a subset $Y \subset X$ is called

(i) compact if and only if for every sequence of elements of Y there exists a subsequence that converges to an element of Y.

(ii) closed if and only if for each sequence $(x_n)_{n\in\mathbb{N}}$ in Y which converges to an element x, we have $x \in Y$.

(2) The b-metric space is complete if every Cauchy sequence converges.

We consider next the following families of subsets of a b-metric space (X,d):

$$P(X) := \{Y \in \mathcal{P}(X) |\ Y \neq \emptyset\};$$

$$P_b(X) := \{Y \in P(X) |\ diam(Y) < \infty\},$$

where

$$diam : P(X) \to \mathbb{R}_+ \cup \{\infty\}, diam(Y) = sup\{d(a,b), a, b \in Y\}$$

is the generalized diameter functional;

$$P_{cp}(X) := \{Y \in P(X) |\ Y \text{ is compact}\};$$

$$P_{cl}(X) := \{Y \in P(X) |\ Y \text{ is closed}\};$$

$$P_{b,cl}(X) := P_b(X) \cap P_{cl}(X)$$

We will introduce the following generalized functionals on a b-metric space (X,d). Some of them were defined in [3].

(1) $D : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$D(A,B) = \inf\{d(a,b) | a \in A, b \in B\},$$

for any $A, B \subset X$.

D is called the gap functional between A and B. In particular, if $x_0 \in X$ then $D(x_0, B) := D(\{x_0\}, B)$.

(2) $\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\delta(A,B) = \sup\{d(a,b) |\ a \in A,\ b \in B\}.$$

(3) $\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\rho(A,B) = \sup\{D(a,B) | a \in A\},$$

for any $A, B \subset X$.

ρ is called the (generalized) excess functional.

(4) $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$H(A,B) = \max\left\{\sup_{x \in A} D(x,B), \sup_{y \in B} D(A,y)\right\},$$

for any $A, B \subset X$.

H is the (generalized) Pompeiu-Hausdorff functional.

Let (X,d) be a b-metric space. If $F : X \to P(X)$ is a multivalued operator, we denote by $Fix(F)$ the fixed point set of F, i.e. $Fix(F) := \{x \in X | x \in F(x)\}$ and by $SFix(F)$ the strict fixed point set of F, i.e. $SFix(F) := \{x \in X | \{x\} = F(x)\}$.

The following results are useful for some of the proofs in the paper.

Lemma 2.1. *(Czerwik* [3]*) Let (X,d) be a b-metric space. Then*

$$D(x,A) \leq s[d(x,y) + D(y,A)], \text{ for all } x, y \in X, A \subset X.$$

Lemma 2.2. *(Czerwik* [3]*) Let (X,d) be a b-metric space and let $\{x_k\}_{k=0}^n \subset X$. Then:*

$$d(x_n, x_0) \leq sd(x_0,x_1) + ... + s^{n-1}d(x_{n-2}, x_{n-1}) + s^{n-1}d(x_{n-1}, x_n).$$

Lemma 2.3. *(Czerwik* [3]*) Let (X,d) be a b-metric space and for all $A, B, C \in X$ we have:*

$$H(A,C) \leq s[H(A,B) + H(B,C)].$$

Lemma 2.4. *(Czerwik* [3]*)*

(1) Let (X, d) be a b-metric space and $A, B \in P_{cl}(X)$. Then for each $\alpha > 0$ and for all $b \in B$ there exists $a \in A$ such that:

$$d(a, b) \leq H(A, B) + \alpha;$$

(2) Let (X, d) be a b-metric space and $A, B \in P_{cp}(X)$. Then for all $b \in B$ there exists $a \in A$ such that:

$$d(a, b) \leq sH(A, B).$$

3. MAIN RESULTS

Theorem 3.1. *Let X be a nonempty set, d and ρ two b-metrics on X with constants $s > 1$ and respectively $t > 1$ and let $F : X \to P(X)$ a multivalued operator. We suppose that:*

(i): *(X, d) is a complete b-metric space;*
(ii): *There exists $c > 0$ such that $d(x, y) \leq c \cdot \rho(x, y)$, for all $x, y \in X$;*
(iii): *$F : (X, d) \to (P(X), H_d)$ is closed;*
(iv): *There exists $0 \leq \alpha < \frac{1}{t}$ such that*

$$H_\rho(F(x), F(y)) \leq \alpha\rho(x, y),$$

for all $x, y \in X$.

Then we have:

(1) $Fix(F) \neq \emptyset$;
(2) For all $x \in X$ and $y \in F(x)$ there exists $(x_n)_{n\in\mathbb{N}}$ such that:
  (a) $x_0 = x, x_1 = y$;
  (b) $x_{n+1} \in F(x_n)$;
  (c) $d(x_n, x^*) \to 0$, as $n \to \infty$, where $x^* \in Fix(F)$;
  (d) $\rho(x_n, x^*) \leq \frac{t\alpha^n}{1-t\alpha}\rho(x_0, x_1)$.

Proof. Let $1 < q < \frac{1}{t\alpha}$ be arbitrary. Take $x_0 \in X$ and for all $x_1 \in F(x_0)$ there exists $x_2 \in F(x_1)$ such that:

$$\rho(x_1, x_2) \leq qH_\rho(F(x_0), F(x_1)) \leq q\alpha\rho(x_0, x_1).$$

For $x_2 \in F(x_1)$ there exists $x_3 \in F(x_2)$ such that:

$$\rho(x_2, x_3) \leq qH_\rho(F(x_1), F(x_2)) \leq q\alpha\rho(x_1, x_2) \leq (q\alpha)^2\rho(x_0, x_1)$$

We can construct by induction a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$\rho(x_n, x_{n+1}) \leq (q\alpha)^n\rho(x_0, x_1), \text{ for all } n \in \mathbb{N}.$$

We will prove next that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy, by estimating $\rho(x_n, x_{n+p})$.

$$\rho(x_n, x_{n+p}) \leq t\rho(x_n, x_{n+1}) + t^2\rho(x_{n+1}, x_{n+2}) + ...+$$

$$+... + t^{p-1}\rho(x_{n+p-2}, x_{n+p-1}) + t^{p-1}\rho(x_{n+p-1}, x_{n+p})$$

$$\leq t(q\alpha)^n\rho(x_0, x_1) + t^2(q\alpha)^{n+1}\rho(x_0, x_1) + ...+$$

$$+... + t^{p-1}(q\alpha)^{n+p-2}\rho(x_0, x_1) + t^{p-1}(q\alpha)^{n+p-1}\rho(x_0, x_1) =$$

$$= t(q\alpha)^n\rho(x_0, x_1)[1 + tq\alpha + ... + (tq\alpha)^{p-2} + t^{p-2}(q\alpha)^{p-1}]$$

$$\leq t(q\alpha)^n\rho(x_0, x_1)[1 + tq\alpha + ... + (tq\alpha)^{p-2} + t^{p-1}(q\alpha)^{p-1}]$$

$$= t(q\alpha)^n\rho(x_0, x_1)\tfrac{1-(tq\alpha)^p}{1-tq\alpha}.$$

But $1 < q < \frac{1}{t\alpha}$ so we obtain that:

$$\rho(x_n, x_{n+p}) \leq t(q\alpha)^n\rho(x_0, x_1)\frac{1 - (tq\alpha)^p}{1 - tq\alpha} \to 0, \tag{3.1}$$

as $n \to \infty$. So $(x_n)_{n\in\mathbb{N}}$ is Cauchy and $x_n \to x \in X$.

From (ii) it follows that the sequence is Cauchy in (X, d). Denote by $x^* \in X$ the limit of the sequence. From (i) and (iii) we get that $d(x_n, x^*) \to 0$, as $n \to \infty$, where $x^* \in FixF$.

In (3.1) we can let $p \to \infty$ and we obtain

$$\rho(x_n, x_{n+p}) \leq \frac{t(q\alpha)^n}{1 - tq\alpha}\rho(x_0, x_1).$$

Making $q \to 1$ we obtain

$$\rho(x_n, x^*) \leq \frac{t\alpha^n}{1 - t\alpha}\rho(x_0, x_1).$$

The proof is complete. □

Theorem 3.2. *Let X be a nonempty set, d and ρ two b-metrics on X with constants $s > 1$ and respectively $t > 1$ and let $F : X \to P(X)$ a multivalued operator. We suppose that:*

(i): *(X, d) is a complete b-metric space;*
(ii): *There exists $c > 0$ such that $d(x, y) \leq c \cdot \rho(x, y)$, for all $x, y \in X$;*
(iii): *$F : (X, d) \to (P(X), H_d)$ is closed;*
(iv): *There exists $0 \leq \alpha < \frac{1}{t}$ such that*

$$H_\rho(F(x), F(y)) \leq \alpha\rho(x, y),$$

for all $x, y \in X$;
(v): *$SFix(F) \neq \emptyset$.*

Then we have:

(1) $Fix(F) = SFix(F) = \{x^*\}$;
(2) $H_\rho(T^n(x), x^*) \leq \alpha^n\rho(x, x^*)$, for all $n \in \mathbb{N}$ and for each $x \in X$;
(3) $\rho(x, x^*) \leq \frac{t}{1-t\alpha}H_\rho(x, F(x))$, for all $x \in X$;

(4) The fixed point problem is well-posed for F with respect to D_ρ and with respect to H_ρ, too.

Proof. 1.-2. We suppose that $x^* \in SFix(F)$. Taking $y = x^*$ in (iv) we obtain $H_\rho(F(x), F(x^*)) = H_\rho(F(x), x^*) \leq \alpha\rho(x, x^*)$, for all $x \in X$.

By induction we have

$$H_\rho(F^n(x), x^*) \leq \alpha^n \rho(x, x^*),$$

for all $x \in X$.

We take now $y^* \in Fix(F), y^* \in F(y^*)$. We have that

$$\rho(y^*, x^*) \leq H_\rho(F^n(y^*), x^*) \leq \alpha^n \rho(y^*, x^*) \to 0,$$

as $n \to \infty$. Hence we have $y^* = x^*$.

3. $\rho(x, x^*) \leq t[H_\rho(x, F(x)) + H_\rho(F(x), x^*)] \leq tH_\rho(x, F(x)) + t\alpha\rho(x, x^*)$.

So we obtain

$$\rho(x, x^*) \leq \frac{t}{1 - t\alpha} H_\rho(x, F(x)).$$

4. Let (x_n) be such that $D_\rho(x_n, F(x_n)) \to 0$, as $n \to \infty$. We will prove that $\rho(x_n, x^*) \to 0$, as $n \to \infty$.

Estimating $\rho(x_n, x^*)$ we have

$$\rho(x_n, x^*) \leq t[\rho(x_n, y_n) + D_\rho(y_n, F(x^*))] \leq t[\rho(x_n, y_n) + H_\rho(F(x_n), F(x^*))],$$

for all $y_n \in F(x_n)$ and for each $n \in \mathbb{N}$.

Taking $\underset{y_n \in F(x_n)}{inf}$ we obtain

$$\rho(x_n, x^*) \leq t[D(x_n, F(x_n)) + H(F(x_n), F(x^*))] \leq tD(x_n, F(x_n)) + t\alpha\rho(, x_n, x^*).$$

Hence we have

$$\rho(x_n, x^*) \leq \frac{t}{1 - t\alpha} D(x_n, F(x_n)) \to 0, \text{ as } n \to \infty.$$

So $x_n \to x^*$. □

We will next give a data dependence result.

Theorem 3.3. *Let X be a nonempty set, d and ρ two b-metrics on X with constants $s > 1$ and respectively $t > 1$ and let $F, T : X \to P(X)$ two multivalued operators. We suppose that:*

(i): *(X, d) is a complete b-metric space;*

(ii): *There exists $c > 0$ such that $d(x, y) \leq c \cdot \rho(x, y)$, for all $x, y \in X$;*

(iii): *$F : (X, d) \to (P(X), H_d)$ is closed;*

(iv): *There exists $0 \leq \alpha < \frac{1}{t}$ such that*

$$H_\rho(F(x), F(y)) \leq \alpha\rho(x, y),$$

for all $x, y \in X$;

(v): *$SFix(F) \neq \emptyset$;*

(vi): *$Fix(T) \neq \emptyset$;*

(vii): *There exists $\eta > 0$ such that $H_\rho(F(x), T(x)) \leq \eta$, for all $x \in X$.*

Then

$$H_\rho(Fix(F), Fix(T)) \leq \frac{t\eta}{1 - t\alpha}.$$

Proof. Let $x^* \in SFix(F)$ and $y^* \in Fix(T)$. We have that

$$\rho(y^*, x^*) \leq H_\rho(T(y^*), x^*) \leq t[H_\rho(T(y^*), F(y^*)) + H_\rho(F(y^*), x^*)]$$
$$\leq t[\eta + H_\rho(F(y^*), F(x^*))] \leq t[\eta + \alpha\rho(y^*, x^*)].$$

Hence we have $\rho(y^*, x^*) \leq \frac{t\eta}{1-t\alpha}$. □

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