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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} anniversary

A modified Seidel method for calculating the fixed points of contractive mappings

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ABSTRACT. For calculating the fixed point of a contractive mapping $\mathbf{f} = (f_1, f_2, ..., f_m) : \Omega \to \Omega,\ \Omega \in \mathbf{R}^m$ we employ a modified Seidel successive approximations scheme:

$$x^{(n)}_{\pi(k)} = f_{\pi(k)}\left(\widetilde{x}^{(n,k)}_1, ..., \widetilde{x}^{(n,k)}_{k-1}, \widetilde{x}^{(n,k)}_k, ..., \widetilde{x}^{(n,k)}_m\right),$$

where

$$\widetilde{x}^{(n,k)}_l = \begin{cases} x^{(n)}_l,\ if\ l = \pi(s)\ and\ s < k, \\ x^{(n-1)}_l,\ if\ l = \pi(s)\ and\ s \geq k. \end{cases}$$

We present a strategy to find the permutation π in order to accelerate the iterative processus.

1. INTRODUCTION

Definition. The function $\mathbf{f} = (f_1, f_2, ..., f_m) : \Omega \to \Omega,\ \Omega \in \mathbf{R}^m$ is a contractive mapping if there exists a constant $c \in (0,1)$ such that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \leq c\,\|\mathbf{x} - \mathbf{y}\|,\ \forall \mathbf{x}, \mathbf{y} \in \Omega, \tag{1.1}$$

We consider on $\mathbf{R}^m$ the norm

$$\|\mathbf{x}\| = \|(x_1, x_2, ..., x_m)\| = |x_1| + |x_1| + ... + |x_1|. \tag{1.2}$$

Definition. $\mathbf{x} \in \Omega$ is a fixed point of the function $\mathbf{f}$ if $\mathbf{f}(\mathbf{x}) = \mathbf{x}$.

We have the following theorem concerning the contraction mappings:

Theorem 1.1. *If$\Omega \in \mathbf{R}^m$ is a closed set and $\mathbf{f} : \Omega \to \Omega$, is a contractive mapping, then it has a unique fixed point $\mathbf{x}$ which can be obtained by means of the successive approximations method starting from an initial approximation $\mathbf{x}^{(0)}$. Hence $\mathbf{x} = \lim_{n\to\infty} \mathbf{x}^{(n)}$ where $\mathbf{x}^{(n)} = \mathbf{f}\left(\mathbf{x}^{(n-1)}\right),\ n = 1, 2, 3, ...$.*

2. THE MODIFIED SEIDEL'S METHOD

First of all we have to notice that generally, when we compute $x^{(n)}_k$ the components $x^{(n)}_1, ..., x^{(n)}_{k-1}$ are already calculated. This observation suggests the following formula, due to Seidel, [3], [4] for calculating the successive approximations:

$$x^{(n)}_k = f_k\left(x^{(n)}_1, ..., x^{(n)}_{k-1}, x^{(n-1)}_k, ..., x^{(n-1)}_m\right),\ k = 2, ..., m. \tag{2.3}$$

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However it is not obligatory (in the framework of the $n-$th iteration) to calculate first $x_1^{(n)}$, then $x_2^{(n)}$ and so on; we may change the order. For instance, for an arbitrary permutation $\pi : \{1, 2, ..., m\} \to \{1, 2, ..., m\}$, we can calculate first $x_{\pi(1)}^{(n)}$, then $x_{\pi(2)}^{(n)}$ (taking into account that $x_{\pi(1)}^{(n)}$ is already calculated) and so on, i.e.

$$x_{\pi(k)}^{(n)} = f_{\pi(k)}\left(\widetilde{x}_1^{(n,k)}, ..., \widetilde{x}_{k-1}^{(n,k)}, \widetilde{x}_k^{(n,k)}, ..., \widetilde{x}_m^{(n,k)}\right), \tag{2.4}$$

where

$$\widetilde{x}_l^{(n,k)} = \begin{cases} x_l^{(n)},\ if\ l = \pi(s)\ and\ s < k, \\ x_l^{(n-1)},\ if\ l = \pi(s)\ and\ s \geq k. \end{cases} \tag{2.5}$$

Our aim is to find the permutation π in order to accelerate the iterative processus.

3. Discretized Hammerstein integral equations

We consider the Hammerstein integral equation

$$x(t) = \int_a^b K(t,s) f(s, x(s))\, ds,\ t \in [a,b]. \tag{3.6}$$

In order to discretize this equation we employ Nyström's method , considering the numeric integration scheme

$$\sum_{j=1}^{m} w_j x(t_j) \approx \int_a^b x(s)\, ds. \tag{3.7}$$

By means of this scheme, from the integral equation (3.6) we get the algebraic system

$$x(t_i) = \sum_{j=1}^{m} w_j K(t_i, t_j) f(t_j, x(t_j)). \tag{3.8}$$

Denoting $x_j = x(t_j), K_{ij} = w_j K(t_i, t_j), f_j = f(t_j, \cdot)$, the system (3.8) becomes

$$x_i = \sum_{j=1}^{m} K_{ij} f_j(x_j). \tag{3.9}$$

Assuming that the mapping defined by (3.9) is a contraction, we consider (according to Seidel's modified method) the sequence of successive approximations

$$x_{\pi(j)}^{(n)} = \sum_{i=1}^{m} K_{\pi(j)j} f_i\left(\widetilde{x}_i^{(n,j)}\right).$$

In order to assess the difference between two consecutive approximations we calculate

$$\sum_{i=1}^{m} \left| x_i^{(n)} - x_i^{(n-1)} \right| = \sum_{j=1}^{m} \left| x_{\pi(j)}^{(n)} - x_{\pi(j)}^{(n-1)} \right| =$$

$$= \sum_{j=1}^{m} \left| \sum_{i=1}^{m} K_{\pi(j)\pi(i)} \left[f_{\pi(i)}\left(\widetilde{x}_{\pi(i)}^{(n,j)}\right) - f_{\pi(i)}\left(\widetilde{x}_{\pi(i)}^{(n-1,j)}\right)\right] \right| \leq$$

$$\leq \sum_{j=1}^{m}\left(\sum_{i=1}^{m}\left|K_{\pi(j)\pi(i)}\right|\right)\left|f_{\pi(i)}\left(\widetilde{x}_{\pi(i)}^{(n,j)}\right)-f_{\pi(i)}\left(\widetilde{x}_{\pi(i)}^{(n-1,j)}\right)\right|. \tag{3.10}$$

The relationship (3.10) is equivalent to

$$\sum_{i=1}^{m}\left|x_i^{(n)}-x_i^{(n-1)}\right|=\sum_{j=1}^{m}\left|x_{\pi(j)}^{(n)}-x_{\pi(j)}^{(n-1)}\right|\leq$$

$$\leq \sum_{j=1}^{m}\left(\sum_{i=1}^{m}\left|K_{\pi(j)\pi(i)}\right|\right)\left|f_{\pi(i)}\left(\widetilde{x}_{\pi(i)}^{(n,j)}\right)-f_{\pi(i)}\left(\widetilde{x}_{\pi(i)}^{(n-1,j)}\right)\right|=$$

$$=\sum_{j=1}^{m}\left(\sum_{i=1}^{j-1}\left|K_{\pi(j)\pi(i)}\right|\left|f_{\pi(i)}\left(x_{\pi(i)}^{(n)}\right)-f_{\pi(i)}\left(x_{\pi(i)}^{(n-1)}\right)\right|+\right.$$

$$\left.+\sum_{i=j}^{m}\left|K_{\pi(j)\pi(i)}\right|\left|f_{\pi(i)}\left(x_{\pi(i)}^{(n-1)}\right)-f_{\pi(i)}\left(x_{\pi(i)}^{(n-2)}\right)\right|\right)=$$

$$=\sum_{j=1}^{m}\sum_{i=1}^{m}\left|K_{\pi(j)\pi(i)}\right|\left|f_{\pi(i)}\left(x_{\pi(i)}^{(n)}\right)-f_{\pi(i)}\left(x_{\pi(i)}^{(n-1)}\right)\right|+$$

$$+\sum_{j=1}^{m}\sum_{i=j}^{m}K_{\pi(j)\pi(i)}\left(\left|f_{\pi(i)}\left(x_{\pi(i)}^{(n-1)}\right)-f_{\pi(i)}\left(x_{\pi(i)}^{(n-2)}\right)\right|-\right.$$

$$\left.-\left|f_{\pi(i)}\left(x_{\pi(i)}^{(n)}\right)-f_{\pi(i)}\left(x_{\pi(i)}^{(n-1)}\right)\right|\right)$$

whence we deduce

$$\sum_{i=1}^{m}\left|x_i^{(n)}-x_i^{(n-1)}\right|\leq\sum_{j=1}^{m}\sum_{i=1}^{m}\left|K_{ji}\right|\left|f_i\left(x_i^{(n)}\right)-f_i\left(x_i^{(n-1)}\right)\right|+$$

$$+\sum_{j=1}^{m}\sqrt{\sum_{i=j}^{m}K_{\pi(j)\pi(i)}^2}\cdot$$

$$\cdot\sqrt{\sum_{i=j}^{m}\left(\left|f_{\pi(i)}\left(x_{\pi(i)}^{(n-1)}\right)-f_{\pi(i)}\left(x_{\pi(i)}^{(n-2)}\right)\right|-\left|f_{\pi(i)}\left(x_{\pi(i)}^{(n)}\right)-f_{\pi(i)}\left(x_{\pi(i)}^{(n-1)}\right)\right|\right)^2}. \tag{3.11}$$

In order to minimize the assessment formula from the right hand side on of the relationship (3.11) we embrace the following selection strategy which leads in the end to the construction of the permutation π :

- $\pi(1)$ is chosen such that

$$\sum_{i=1}^{m}K_{\pi(1)\pi(i)}^2=\sum_{i=1}^{m}K_{\pi(1)i}^2=\min_{j\in\{1,\ldots,m\}}\sum_{i=1}^{m}K_{ji}^2, \tag{3.12}$$

- $\pi(2)$ is chosen such that

$$\sum_{i=2}^{m} K_{\pi(2)\pi(i)}^{2} = \sum_{i\in\{1,...,m\}\setminus\{\pi(1)\}} K_{\pi(2)i}^{2} = \min_{j\in\{1,...,m\}\setminus\{\pi(1)\}} \sum_{i\in\{1,...,m\}\setminus\{\pi(1)\}} K_{ji}^{2}, \tag{3.13}$$

- $\pi(k)$, $k = 3, 4, ..., m$ is chosen such that

$$\sum_{i=k}^{m} K_{\pi(k)\pi(i)}^{2} = \sum_{i\in\{1,...,m\}\setminus\{\pi(1),\pi(2),...,\pi(k-1)\}} K_{\pi(k)i}^{2}$$

$$= \min_{j\in\{1,...,m\}\setminus\{\pi(1),\pi(2),...,\pi(k-1)\}} \sum_{i\in\{1,...,m\}\setminus\{\pi(1),\pi(2),...,\pi(k-1)\}} K_{ji}^{2}. \tag{3.14}$$

4. THE SELECTION PROCEDURE .

We achieve the selection procedure described above as follows: we consider the vector $\mathbf{A} = (A(1), A(2), ..., A(m))$ with $A(l) = \sum_{s=1}^{m} K_{ls}^{2} + il$, $i = \sqrt{-1}$. We organize the vector $\mathbf{A}$ as a heap [1] imposing $Real(A(l)) < Real(A(2l))$ si $Real(A(l)) < Real(A(2l+1))$ and we employ the following algorithm (written in pseudocode) for constructing the permutation π:

Procedure **ConstructPermutation**:

$length \leftarrow m$
$k \leftarrow 1$
While $length > 0$ **do**
$\{$**BuildHeap**$(\mathbf{A})$
$\pi(k) = imag(A(1))$
$p = 1$
While $p < length$ **do**
$\{A(p) = A(p+1) - K_{\pi(k)p}^{2}$
$p = p + 1\}$
$A(length) \leftarrow \infty$
$length \leftarrow length - 1$
$k \leftarrow m - length + 1\}$

Another procedure deals with the initial values of $\mathbf{x}$ and $\mathbf{A}$:

Procedure **Initialization**

For $l \leftarrow 1...m$
$\{x[l] \leftarrow 0;\ y[l] \leftarrow 1;$
$error = \sum_{l=1}^{m} |x[l] - y(l)|;$
$A(l) = \sum_{s=1}^{m} K_{ls}^{2} + il\}$

The last procedure deals with the computation of the fixed point

Procedure **FixedPoint**

While $error > \varepsilon$ **do**
$\{$**for** $l = 1...m$
$\{y[\pi(l)] \leftarrow x[\pi(l)]$
$x[\pi(l)] \leftarrow \sum_{j=1}^{m} K_{\pi(l)\pi(j)} f_{\pi(j)}\left(x_{\pi(j)}\right)\}$
$error \leftarrow \sum_{l=1}^{m} |x[l] - y(l)|;\}$

For using the modified Seidel successive approximation method for solving the algebraic system (which is assumed to be a contraction)

$$x_i = \sum_{j=1}^{m} K_{ij} f_j(x_j), \ i = 1, ..., m, \tag{4.15}$$

we utilize the following code
Initialization
ConstructPermutation
FixedPoint

5. Application

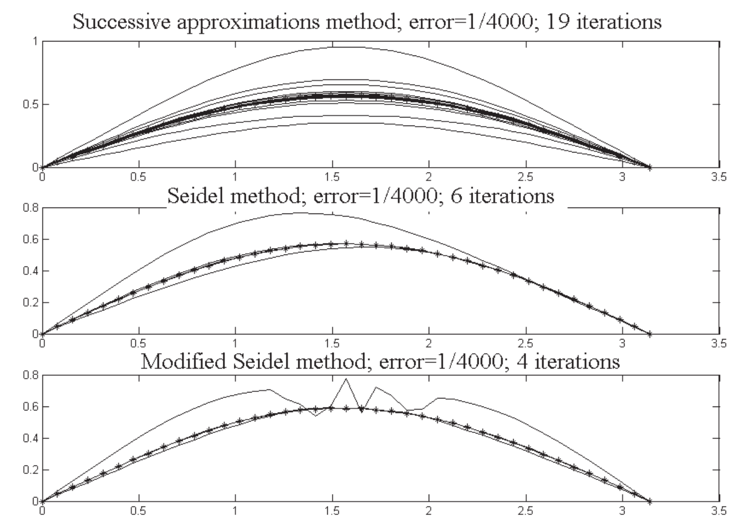

Figure 1. Comparison of the three methods

We shall consider the Hammerstein equation that appears in the study of the free-surface flow past a circular obstacle [2]

$$x(t) = \frac{\lambda}{\pi} \int_0^{\pi} \exp(-x(s)) \ln \left| \frac{\sin \frac{t+s}{2}}{\sin \frac{t-s}{2}} \right| (1 + \sin s) \sin s ds. \tag{5.16}$$

Since we have an integrable logarithmic singularity, in order to isolate the singularity, we shall write the integral equation as follows

$$x(t) = \frac{\lambda}{\pi} \int_0^{\pi} \exp(-x(s)) \ln \left| \frac{(t-s) \sin \frac{t+s}{2}}{\sin \frac{t-s}{2}} \right| (1 + \sin s) \sin s ds -$$

$$-\frac{\lambda}{\pi}\int_0^{\pi}\left[\exp\left(-x\left(s\right)\right)\left(1+\sin s\right)\sin s-\exp\left(-x\left(t\right)\right)\left(1+\sin t\right)\sin t\right]\ln\left|t-s\right|ds-$$
$$-\frac{\lambda}{\pi}\exp\left(-x\left(t\right)\right)\left(1+\sin t\right)\sin t\int_0^{\pi}\ln\left|t-s\right|ds.$$

We consider on the segment $[0,\pi]$ a grid consisting of the nodes $\{t_0,t_1,...,t_n\}$ with $t_i=\dfrac{i}{n}\pi,\ i=0,1,...,n$. Employing the trapeziums formula

$$\int_0^{\pi}f\left(s\right)ds=\frac{\pi}{2n}\left[f\left(t_0\right)+2\sum_{i=1}^{n-1}f\left(t_i\right)+f\left(t_n\right)\right]$$

and taking into account that $\sin t_0=\sin t_n=0$, we obtain

$$x\left(t_i\right)=\sum_{j=1}^{n-1}K_{ij}\exp\left(-x\left(t_j\right)\right),i=0,1,...,n, \tag{5.17}$$

with

$$K_{ij}=\frac{\lambda}{n}\ln\left|\frac{\sin\frac{t_i+t_j}{2}}{\sin\frac{t_i-t_j}{2}}\right|\left(1+\sin t_j\right)\sin t_j,\ i\neq j,$$
$$K_{jj}=\lambda\left(1+\sin t_j\right)\sin t_j\cdot\left[\frac{\ln\left|4\sin^2 t_j\left(t_0-t_j\right)\left(t_n-t_j\right)\right|}{2n}+\right.$$
$$\left.+\sum_{i=1,i\neq j}^{n-1}\frac{\ln\left|t_i-t_j\right|}{n}-\frac{\pi-t_j}{\pi}\ln\left(\pi-t_j\right)-\frac{t_j}{\pi}\ln t_j+1\right].$$

In [2] one demonstrates that for $\lambda=1/2$, the mapping defined by (5.17) is a contraction

In Figure 1 we present by stars (*) the solution of the system obtained by the successive approximations method (19 iterations), by Seidel's method (6 iterations) and by the modified Seidel method (4 iterations). We used continuous lines for presenting the intermediate approximations.

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