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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} anniversary

Some applications of the fixed-point theory in economics

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ABSTRACT. In this paper, firstly we prove some fixed-point results and then we apply these results in economics, giving two general equilibrium theorems. These theorems give sufficient conditions for the existence of an equilibrium point (and a maximal element) for a generalized abstract economy (respectively for a qualitative game).

1. INTRODUCTION

Many results concerning the fixed-point theory can be applied in the equilibrium theory, giving, for example, the existence of a solution for the equilibrium in the abstract economies or generalized games and in the generalized abstract economies with preference multimaps.

In 1975, W. Shafer and H. Sonnenschein proved the existence of the equilibrium for abstract economies without ordered preferences.Over the last forty years, more general existence results appeared in the literature (for example: A. Borglin and H. Keiding (1976), D. Gale and A. Mas Colell (1978), N. C. Yannelis (1987), C. Ionescu-Tulcea (1988), E. Tarafdar (1988), K. K. Tan and G. X.- Z. Yuan (1994)). All these results assume directly or indirectly the lower-semicontinuity of the multimaps representing the constraints of each agent. In 1999, G. X.- Z. Yuan and E. Tarafdar proved some existence theorems for the equilibrium of the compact or noncompact qualitative games and generalized games in which the constraint or the preference multimaps have supplementary properties. The existence of the equilibrium in an abstract economy with compact strategy sets in $\mathbb{R}^n$ was proved in a seminal paper by G. Debreu. The Debreu's theorem extended the earlier work of J. Nash in the game theory and have many generalizations, for example, by A. Borglin and H. Keiding (1976). Following their paper and, also, the paper of D. Gale and A. Mas-Colell (1978), on non-ordered preference relations, many theorems on the existence of maximal elements of the preference relations, which may not be transitive or complete have been proved by T. C. Bergs (1976), M. Walker (1977), N. C. Yannelis and D. Prabhakar (1983), S. Toussaint (1984), N. C. Yannelis (1985), C. Ionescu-Tulcea (1988), G. Mehta (1990).

In this paper we give two results of Tarafdar type equilibrium theorems, more precisely, an equilibrium theorem for the generalized abstract economies and a maximal element theorem for the qualitative games. For the proof of the existence

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of an equilibrium point, we use some results, presented without proofs in [3], about the existence of a fixed-point for a family of multimaps, giving now their proofs.

Firstly, we recall **some definitions and notations**, reviewing the mathematical and economical concepts that we need. For a nonempty set X, 2^X denotes the class of all subsets of X.

Definition 1.1. A *multimap* is a function $T : X \to 2^Y$; in another terminology, a multimap is also known as a *set valued function*, a *mapping*, a *map* or a *correspondence* (here X and Y are two nonempty sets).

A multimap $T : X \to 2^Y$ is *nonempty-valued* (*convex-valued*, or *compact-valued*) if the set $T(x)$ is a *nonempty* (respectively *convex*, or *compact*) for each $x \in X$.The *fiber* of the multimap $T : X \to 2^Y$ at the point $y \in Y$, is the set $T^{-1}(y) = \{x \in X : y \in T(x)\}$.

Definition 1.2. Let I be a *countable* or *uncountable* set of *agents* (or *players).* For each $i \in I$, suppose her/his *choice* or *strategy* set X_i is a nonempty subset of a topological vector space. Let $X = \prod_{i\in I} X_i$. For each $i \in I$, let $P_i : X \to 2^{X_i}$ be a multimap. Following the notion of D. Gale and A. Mas-Colell, the collection $\Gamma = (X_i, P_i)_{i\in I}$ is called a *qualitative game.*An element $\widetilde{x} \in X$ is said to be a *maximal point* of the game Γ , if $P_i(\widetilde{x}) = \emptyset$, for all $i \in I$.

Definition 1.3. A *generalized abstract economy* (or a *generalized game*) is a family of quadruples $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$, where I is a (finite or infinite) set of agents (players) such that, for each $i \in I$, X_i is a nonempty subset of a topological vector space; $A_i, B_i : X = \prod_{i\in I} X_i \to 2^{X_i}$ are constraint multimaps and $P_i : X \to 2^{X_i}$ is a preference multimap. An *equilibrium* for Γ is a point $\widetilde{x} = (\widetilde{x}_i)_{i\in I} \in X$ such that, for each $i \in I$, $\widetilde{x}_i \in B_i(\widetilde{x})$ and $A_i(\widetilde{x}) \cap P_i(\widetilde{x}) = \emptyset$.

2. PRELIMINARIES

The following proposition (see [3]), has a starting point a result of [2], generalized in a certain sense in [4]. We notice that in the last two mentioned results, appeared two families of multimaps.

Let I be an index set and for each $i \in I$, let E_i be a Hausdorff topological vector space. Let $(X_i)_{i\in I}$ be a family of nonempty convex subsets with each X_i in E_i. Let $X = \prod_{i\in I} X_i$. Let also $C \subseteq X$ be a nonempty compact subset.

Proposition 2.1. *For each $i \in I$, let $T_i : X \to 2^{X_i}$ be a nonempty-valued and convex-valued multimap. Suppose that the following conditions hold:*

(1) for each $i \in I$, X can be covered with the interiors of all fibers of T_i, i.e.

$$X = \bigcup \{int_X T_i^{-1}(y_i) :\ y_i \in X_i\};$$

(2) if X is not compact, assume that for each $i \in I$ and for each finite subset F_i of X_i, there exists a nonempty compact convex set C_{F_i} in X_i such that $C_{F_i} \supseteq F_i$ and $X \setminus C$ can be covered with the interiors of all fibers of T_i at the points of C_{F_i}, i.e.

$$X \setminus C \subseteq \bigcup \{int_X T_i^{-1}(y_i) :\ y_i \in C_{F_i}\}.$$

Then, there exists $\widetilde{x} = (\widetilde{x}_i)_{i\in I} \in X$, such that $\widetilde{x}_i \in T_i(\widetilde{x})$, for each $i \in I$ (i.e. $\widetilde{x}$ is a fixed-point for the family $(T_i)_{i\in I}$).

Proof. From the hypothesis *(1)*, because the set C is compact, it follows that, for each $i \in I$, there exists a finite set $F_i \subset X_i$ such that

$$C \subseteq \bigcup_{y_i\in F_i} intT_i^{-1}(y_i). \tag{2.1}$$

From *(2)*, it follows the existence of a compact convex set $C_{F_i} \subseteq X_i$ such that $C_{F_i} \supseteq F_i$ and

$$X \setminus C \subseteq \bigcup_{y_i\in C_{F_i}} intT_i^{-1}(y_i). \tag{2.2}$$

Let $F = \prod_{i\in I} F_i$ (a finite set in X) and $C_F = \prod_{i\in I} C_{F_i}$ (a compact convex set in X). It follows that $C_F \supseteq F$ and $C_F \setminus C \subseteq X \setminus C$. Now, using (2.2), we obtain the following inclusion:

$$C_F \setminus C \subseteq \bigcup_{y_i\in C_{F_i}} intT_i^{-1}(y_i). \tag{2.3}$$

Since $F_i \subseteq C_{F_i}$, from (2.1) and (2.3) we have $C_F \subseteq \bigcup_{y_i\in C_{F_i}} intT_i^{-1}(y_i)$. But the set C_F is compact. Therefore, it follows that: for each $i \in I$, there exists a finite set $Y_i = \{y_{i,1}, \ldots, y_{i,m_i+1}\} \subseteq C_{F_i}$ such that $C_F \subseteq \bigcup_{j=1}^{m_i+1} intT_i^{-1}(y_{i,j})$.

Because C_F is compact, then also exists a continuous partition of unity $\{\lambda_{i,1}, \ldots ,\lambda_{i,m_i+1}\}$ subordinated to the open covering $(intT_i^{-1}(y_{i,j}))_{j=1}^{m_i+1}$, that is, for each $j \in \{1, 2, \ldots, m_i + 1\}$, $\lambda_{i,j} : C_F \to [0, 1]$ is a continuous function such that, for all $x \in C_F$, $\sum_{j=1}^{m_i+1} \lambda_{i,j}(x) = 1$ and $\lambda_{i,j}(x) = 0$, for all $x \notin intT_i^{-1}(y_{i,j})$.

In other words, $\lambda_{i,j}(x) \neq 0$ if and only if $x \in intT_i^{-1}(y_{i,j}) \subseteq T_i^{-1}(y_{i,j})$, that is $y_{i,j} \in T_i(x)$, for each $j = 1, \ldots, m_i + 1$ and each $i \in I$.

We consider in $\mathbb{R}^{m_i+1}$, the standard m_i-simplex Δ_{m_i} with vertices $e_{i,1}, \ldots, e_{i,m_i+1}$, each $e_{i,j}$ being the j^{th} unit vector of $\mathbb{R}^{m_i+1}$, hence $\Delta_{m_i} = co(e_{i,1}, \ldots, e_{i,m_i+1})$. Consider also the continuous function $f_i : C_F \to \Delta_{m_i}$ defined by

$$f_i(x) := \textstyle\sum_{j=1}^{m_i+1} \lambda_{i,j}(x)e_{i,j}, \text{ for each } x \in C_F.$$

Now, for each $i \in I$ and $Y_i = \{y_{i,1}, \ldots, y_{i,m_i+1}\}$ we define the map $g_i : \Delta_{m_i} \to co(Y_i) \subseteq C_{F_i}$, by

$$g_i\Big(\sum_{j=1}^{m_i+1} \mu_{i,j}e_{i,j}\Big) := \sum_{j=1}^{m_i+1} \mu_{i,j}y_{i,j},$$

where $\mu_{i,j} \geq 0$, for each $i \in I$ and $j \in \{1, 2, \ldots, m_i + 1\}$, and $\sum_{j=1}^{m_i+1} \mu_{i,j} = 1$. Then, for each $i \in I$, it follows that g_i is continuous.

Let $J_i(x)$ be the set $\{j \in \{1, 2, \ldots, m_i + 1\} : \lambda_{i,j}(x) \neq 0\}$. For each $x \in C_F$, we have: $(g_i \circ f_i)(x) = g_i(\sum_{j=1}^{m_i+1} \lambda_{i,j}(x)e_{i,j}) = \sum_{j=1}^{m_i+1} \lambda_{i,j}(x)y_{i,j} = \sum_{j\in J_i(x)} \lambda_{i,j}(x)y_{i,j} \in coT_i(x) = T_i(x)$, because T_i is convex-valued. Hence $(g_i \circ f_i)(x) \in T_i|_{C_F}(x)$.

Now, we consider $Z_i = span(\Delta_{m_i})$, $i \in I$, $Z = \prod_{i\in I} Z_i$, $K = \prod_{i\in I} \Delta_{m_i}$ (a compact convex set in Z), and the (continuous) maps $\varphi : K \to C_F$ and $\psi : C_F \to K$, defined by $\varphi((z_i)_{i\in I}) := (g_i(z_i))_{i\in I}$ and $\psi(x) := (f_i(x))_{i\in I}$ (for $x \in C_F$). Then, applying the Tychonoff's fixed-point theorem, see [5], to the (well-defined and continuous) map $\theta : K \to K$, $\theta := \psi \circ \varphi$, there exists an element $\widetilde{z} = (\widetilde{z}_i)_{i\in I} \in K$ such that $\widetilde{z} = \theta(\widetilde{z})$. Denoting $\varphi(\widetilde{z}) = \widetilde{x} \in X$, it follows that $\widetilde{z} = \psi(\widetilde{x})$. Hence, for each $i \in I$, $\widetilde{z}_i = f_i(\widetilde{x})$ and $\widetilde{x}_i = g_i(\widetilde{z}_i) = g_i(f_i(\widetilde{x})) \in T_i(\widetilde{x})$. □

The following collectively fixed-point result (see for example [3]) can be proved with understanding changes in the proof of the Proposition 2.1 and generalizes Theorem 1 of [2].

Proposition 2.2. *For each $i \in I$, let $S_i, T_i : X \to 2^{X_i}$ be two nonempty-valued multimaps, such that:*

(0) for each $i \in I$ and each $x \in X$, $coS_i(x) \subseteq T_i(x)$;

(1) for each $i \in I$, X can be covered with the interiors of all fibers of S_i, that is

$$X = \bigcup\{int_X S_i^{-1}(y_i) :\ y_i \in X_i\};$$

(2) if X is not compact, assume that for each $i \in I$ and for each finite subset F_i of X_i, there exists a nonempty compact convex set C_{F_i} in X_i such that $C_{F_i} \supseteq F_i$ and

$$X \setminus C \subseteq \bigcup\{int_X S_i^{-1}(y_i) :\ y_i \in C_{F_i}\}.$$

Then, there exists $\widetilde{x} = (\widetilde{x}_i)_{i\in I} \in X$, such that $\widetilde{x}_i \in T_i(\widetilde{x})$, for each $i \in I$.

Remark 2.1. Obviously, according to the condition "*(0)*", only S_i must be a nonempty-valued multimap. As a simple consequence of the Proposition 2.2, we obtain the following result (see [2], Theorem 1).

Corollary 2.1. *For each $i \in I$, let $S_i, T_i : X \to 2^{X_i}$ be two nonempty-valued multimaps, such that:*

(0) for each $i \in I$ and each $x \in X$, $coS_i(x) \subseteq T_i(x)$;

(1) for each $i \in I$, X can be covered with the interiors of all fibers of S_i, that is

$$X = \bigcup\{int_X S_i^{-1}(y_i) :\ y_i \in X_i\};$$

(2) if X is not compact, assume that for each $i \in I$, there exists a nonempty compact convex subset C_i of X_i such that $X \setminus C \subseteq \bigcup\{int_X S_i^{-1}(y_i) :\ y_i \in C_i\}$.

Then, there exists $\widetilde{x} = (\widetilde{x}_i)_{i\in I} \in X$, such that $\widetilde{x}_i \in T_i(\widetilde{x})$, for each $i \in I$.

Proof. For each $i \in I$ and each finite subset F_i of X_i, we define $C_{F_i} = co(C_i \cup F_i)$. Therefore, it follows that $C_{F_i} \supseteq F_i$ and the set C_{F_i} is compact and convex, according, for example to [1, Corollary 5.30]. Now, we apply Proposition 2.2. □

3. Main results

In this section, we will apply Corollary 2.1 to the existence of the *equilibrium points* (and *maximal elements*) for general abstract economies (respectively qualitative games). The *index set I* will be any set (countable or not) of agents (or players). The *choice set* (the *strategy set*) X_i will be the nonempty set of actions available to the agent i, and for each $x \in X$ and $i \in I$, $A_i(x)$ (respectively $B_i(x)$) will be the state attainable for the agent i , at x , under the constraint A_i (respectively B_i) and $P_i(x)$ is the state preference by the agent i, at x.

In the following theorem, we will consider $X = \prod_{i \in I} X_i$ and $K = \prod_{i \in I} K_i$, where, for each $i \in I$, K_i is a nonempty compact subset of X_i (and hence K is a nonempty compact subset of X).

Theorem 3.1. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be a generalized abstract economy (or a generalized game) such that:*

(1) for each $i \in I$, X_i is a nonempty convex subset of a locally convex topological vector space E_i and K_i is a nonempty compact subset of X_i, such that $\overline{co}K_i$ lies in a complete metrizable subset of E_i;
(2) for each $i \in I$ and $x \in X$, $coA_i(x) \subseteq B_i(x) \subseteq K_i$ and $P_i(x) \subseteq K_i$;
(3) for each $i \in I$, $X = \bigcup \{int_X(A_i^{-1}(y_i) \cap (P_i^{-1}(y_i) \cup G_i)) : \ y_i \in K_i\}$, where $G_i = \{x \in X : A_i(x) \cap P_i(x) = \emptyset\}$;
(4) for each $x = (x_i)_{i \in I} \in X$, $x_i \notin coP_i(x)$.

Then Γ has an equilibrium point in X, that is there exists $\widetilde{x} \in X$ such that for each $i \in I$, $\widetilde{x}_i \in B_i(\widetilde{x})$ and $A_i(\widetilde{x}) \cap P_i(\widetilde{x}) = \emptyset$ (hence $\widetilde{x}_i \in G_i$).

Proof. For each $i \in I$, we consider the following sets and multimaps: $F_i = \{x \in X : \ A_i(x) \cap P_i(x) \neq \emptyset\}$, and $S_i, T_i : X \to 2^{X_i}$ defined by:

$$S_i(x) = \begin{cases} A_i(x) \cap coP_i(x) & , x \in F_i \\ A_i(x) & , x \in G_i \end{cases} ,$$

$$T_i(x) = \begin{cases} B_i(x) \cap coP_i(x) & , x \in F_i \\ B_i(x) & , x \in G_i \end{cases} .$$

Then we can prove that the hypothesis of Corollary 2.1 are fulfilled. Indeed, *the condition (0) of Corollary* 2.1 *holds*, i.e. for each $i \in I$ and $x \in X$, $coS_i(x) \subseteq T_i(x)$. Let $x_i \in coS_i(x)$. It follows that $x_i = \sum_{j=1}^{n} \alpha_j x_{i,j}$ with $\alpha_j > 0$, $\sum_{j=1}^{n} \alpha_j = 1$ and $x_{i,j} \in S_i(x)$, for all $j = \overline{1,n}$.

Case 1. If $x \in F_i$, then for each $j = \overline{1,n}$, we have $x_{i,j} \in A_i(x) \cap coP_i(x)$ and therefore $x_i \in coA_i(x) \subseteq B_i(x)$ and $x_i \in coP_i(x)$, that is $x_i \in B_i(x) \cap coP_i(x) = T_i(x)$.

Case 2. If $x \in G_i$, then for each $j = \overline{1,n}$, we have $x_{i,j} \in S_i(x) = A_i(x)$ and hence $x_i \in coA_i(x) \subseteq B_i(x)$. So, $x_i \in T_i(x)$.

Now, let us prove that *the condition (1) of Corollary* 2.1 *holds*, even for K_i instead of X_i, that is $X = \bigcup \{intS_i^{-1}(y_i) : \ y_i \in K_i\}$. We remark that for each $i \in I$ and $y_i \in K_i$, we have:

$$S_i^{-1}(y_i) = (A_i^{-1}(y_i) \cap (coP_i)^{-1}(y_i) \cap F_i) \cup (A_i^{-1}(y_i) \cap G_i). \qquad \text{(I)}$$

Indeed, if $x \in S_i^{-1}(y_i)$ then $y_i \in S_i(x)$ and hence $y_i \in A_i(x) \cap coP_i(x)$, if $x \in F_i$ or $y_i \in A_i(x)$, if $x \in G_i$. Then, $x \in A_i^{-1}(y_i)$ and $x \in (coP_i)^{-1}(y_i)$ and $x \in F_i$, or $x \in A_i^{-1}(y_i)$ and $x \in G_i$, hence the equality (I) is valid.

But $(coP_i)^{-1}(y_i) \supseteq P_i^{-1}(y_i)$, hence, from (I) we obtain:

$$S_i^{-1}(y_i) \supseteq (A_i^{-1}(y_i) \cap P_i^{-1}(y_i) \cap F_i) \cup (A_i^{-1}(y_i) \cap G_i) =$$
$$= (A_i^{-1}(y_i) \cap P_i^{-1}(y_i)) \cup (A_i^{-1}(y_i) \cap G_i) = A_i^{-1}(y_i) \cap (P_i^{-1}(y_i) \cup G_i). \quad \text{(II)}$$

Now, if $x \in X$, then from the hypothesis *(3)*, it follows that for each $i \in I$, there exists $y_i \in K_i$ such that $x \in int(A_i^{-1}(y_i) \cap (P_i^{-1}(y_i) \cup G_i)) \subseteq^{(\mathrm{II})} intS_i^{-1}(y_i)$, hence $x \in \bigcup\{intS_i^{-1}(y_i) : \ y_i \in K_i\}$.

Finally, we remark that *the condition (2) of Corollary* 2.1 *holds*, that is for each $i \in I$, there exists a nonempty compact convex subset C_i of X_i such that $X \setminus K \subseteq \bigcup\{intS_i^{-1}(y_i) : \ y_i \in C_i\}$. We apply our hypothesis *(3)*, putting $C_i = \overline{co}K_i$, which is a compact set, according to [1, Theorem 5.35]. Then, we can apply Corollary 2.1, obtaining the existence of an element $\widetilde{x} = (\widetilde{x}_i)_{i\in I} \in X$, such that $\widetilde{x}_i \in T_i(\widetilde{x})$, for each $i \in I$, hence: $\widetilde{x} \in F_i$ and $\widetilde{x}_i \in B_i(\widetilde{x}) \cap coP_i(\widetilde{x})$ or $\widetilde{x} \in G_i$ and $\widetilde{x}_i \in B_i(\widetilde{x})$. But from the hypothesis *(4)*, $\widetilde{x}_i \notin coP_i(\widetilde{x})$. Then, it follows that $\widetilde{x}_i \in B_i(\widetilde{x})$ and $\widetilde{x} \in G_i$, that is $A_i(\widetilde{x}) \cap P_i(\widetilde{x}) = \emptyset$. □

The following result, a maximal element theorem for a qualitative game, is actually a consequence of our Theorem 3.1.

Theorem 3.2. *Let $\Gamma = (X_i, P_i)_{i\in I}$ be a qualitative game, $X = \prod_{i\in I} X_i$ and suppose that:*

- *(1) for each $i \in I$, X_i is a nonempty convex subset of a locally convex topological vector space E_i, and K_i is a nonempty compact subset of X_i, such that $\overline{co}K_i$ lies in a complete metrizable subset of E_i and $P_i : X \to 2^{K_i}$;*
- *(2) for each $i \in I$, $X = \bigcup\{int_X(P_i^{-1}(y_i) \cup G_i) : \ y_i \in K_i\}$, where $G_i = \{x \in X : P_i(x) = \emptyset\}$;*
- *(3) for each $x = (x_i)_{i\in I} \in X$, $x_i \notin coP_i(x)$.*

Then, Γ has a maximal element (an equilibrium point) in X, that is, there exists $\widetilde{x} \in X$ such that $P_i(\widetilde{x}) = \emptyset$, for each $i \in I$.

Proof. For each $i \in I$, we define the constraint multimaps $A_i, B_i : X \to 2^{X_i}$ by $A_i(x) = B_i(x) = K_i$ for all $x \in X$. It follows that for each $i \in I$ and $y_i \in K_i$, $A_i^{-1}(y_i) = X$ (for each $x \in X$, $x \in A_i^{-1}(y_i)$ because $y_i \in A_i(x) = K_i$).

Therefore, the conditions *(2)* and *(3)* of Theorem 3.1 are fulfilled. Indeed, for example, *(2)* of Theorem 3.2 implies *(3)* of Theorem 3.1, because

$$X = \bigcup\{int_X(P_i^{-1}(y_i) \cup G_i) : \ y_i \in K_i\} = \bigcup\{int(X \cap (P_i^{-1}(y_i) \cup G_i)) : y_i \in K_i\} =$$

$$= \bigcup\{int(A_i^{-1}(y_i) \cap (P_i^{-1}(y_i) \cup G_i)) : \ y_i \in K_i\}.$$

Applying Theorem 3.1, we find $\widetilde{x} \in X$ such that, for each $i \in I$, $\widetilde{x}_i \in B_i(\widetilde{x}) = K_i$ and $K_i \cap P_i(\widetilde{x}) = A_i(\widetilde{x}) \cap P_i(\widetilde{x}) = \emptyset$.

But $P_i : X \to 2^{X_i}$. If $\widetilde{x} \in X$ is such that $P_i(\widetilde{x}) \nsubseteq K_i$, then $\widetilde{x} \notin P_i^{-1}(y_i)$ for each $y_i \in K_i$ and according to the hypothesis *(2)* it follows that $P_i(\widetilde{x}) = \emptyset$. If $\widetilde{x} \in X$ is such that $P_i(\widetilde{x}) \subseteq K_i$, then $\emptyset = K_i \cap P_i(\widetilde{x}) = P_i(\widetilde{x})$. Hence, certainly, $P_i(\widetilde{x}) = \emptyset$. Therefore, $\widetilde{x} \in X$ is a *maximal point* of the game $(X_i, P_i)_{i\in I}$. □

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