CREATIVE MATH. & INF. **Online version at** `http://creative-mathematics.ubm.ro/`
17 (2008), No. 3, 399 - 405 **Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X**

Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} *anniversary*

Numerical investigations of the dynamic problem in thin thermoelastic plates

REMUS ENE AND IOANA DRAGOMIRESCU

ABSTRACT. An operatorial approach for the equations obtained by the homogenization method in the case of thin thermoelastic porous plates is introduced. Based on this formulation an uniqueness theorem concerning the solution is deduced. The dynamical equations obtained by the homogenization method for thin thermoelastic porous plates are numerically investigated showing a stabilization of the solution.

1. INTRODUCTION

The new interest in thin thermoelastic plates is motivated in part by the observation that thermoelasticity can be in certain circumstances a dominant and unavoidable source of dissipation and noise. The two dimensional nature of thin thermoelastic plates introduce flexure in two directions which imply 3-dimensional mathematical models and more complicated analytical and numerical investigations [7]. This complexity is in fact main reason for the the derivation of simpler governing equations including thermal effects (thermal stress, strain and deformation) and similar to the Kirchhoff equations [2], [10]. In [3], following the theory of Lord-Shulman [8] and Green-Lindsay [6] for thermoelastic plates, numerical evaluations on the thermal stresses in a thin porous plate due to the radiations of a thermal source are given.

Here our main interest is to obtain an uniqueness result for the solution of the corresponding dynamical problem. The mathematical model governing the thermoelasticity of the homogeneous plate is obtained using the homogenization method. The uniqueness theorem is based on the following result: the conduction-convection tensor is a positive definite operator. The design of structures operating at an elevated temperature such as nuclear operators, chemical plants are a serious motivation for the investigation of plane thermal stresses in a multiply-connected region. In [12] the proposed method for the solution of the plane thermoelastic problem is based on the necessary integral conditions for the single-valuedness of solution and displacement. In our case, the temperature and the displacement field are used to clarify numerically some stability aspects of the solution.

Received: 21.12.2008. In revised form: 15.01.2009. Accepted: 11.05.2009.
2000 *Mathematics Subject Classification*. 74A10, 74A15.
Key words and phrases. *Thermoelasticity, porous plates, thermoelastic plates, homogenization method.*

2. THE MATHEMATICAL MODEL

Let us consider the following Dirichlet problem [9]

$$\mathcal{L}(\mathbf{x}, D)u = f, \tag{2.1}$$
$$u|\partial\Omega = 0. \tag{2.2}$$

In our particular case, the equation (2.1) defines the governing dynamical equation of the homogeneous and isotropic media and it is given by

$$\sum_{j=1}^{3} \frac{\partial}{\partial x_j}\Big(\sum_{k,l=1}^{3} a_{ijkl} \cdot e_{kl} - \beta\theta\delta_{ij}\Big) = \rho_0 \frac{\partial^2 u_i}{\partial t^2}, \;\; i = 1, 2, 3.$$

In (2.1)-(2.2), the equation is the divergence equation, so the entire problem is called the homogeneous Dirichlet problem for the divergence equation.

Due to a relaxation condition concerning the smoothness of the solution of the problem (2.1)-(2.2), in [9] some generalized solutions are obtained. This type of solutions are very useful not only in technical and physical applications but also for other branches in mathematics. The uniqueness and existence theorems of the generalized solution of the Dirichlet (or Neumann) divergence problem deduced in [9] are based on the existence of eigenvalues and corresponding eigenvectors. Conditions for the existence of eigenvalues and eigenvectors are also given.

Nomenclature

$t-$ time ;
$\beta -$ a constitutive material coefficient ;
$c_t -$ the heat transfer coefficient ;
$S^* -$ the heat source, assumed constant;
$a_{ijkl}^H -$ the elasticity tensor;
$a_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$;
$\mathcal{T} = [0, \;\; t_1] -$ time interval, $\;t_1 > 0$;
$\Omega = [0, L] \times [0, l] \times [-\dfrac{h_0}{2}, \dfrac{h_0}{2}] -$ the plate domain ;
$\delta_{ij} -$ the Kronecker symbol;
$\theta^0 -$ the main term in the asymptotic expression of the absolute temperature of the plate given by the homogenization method [11];
$\rho_0 -$ the density value at $t = 0$ assumed constant ;
$\sigma_{ij}^H -$ the stress tensor given by the homogenization method;

$\overline{x} -$ the coordinate vector of a point in $\Omega \subset \mathbb{R}^3$;
$c_e -$ the specific heat ;
$k -$ the conduction-convection coefficient ;
$\sigma -$ the Stefan-Boltzmann constant;
$L, \; l, \; h_0 -$ the dimensions of the plate ;
$\lambda, \mu -$ Lam$\acute{e}$ constants;
$\Sigma^+, \Sigma^- -$ upper and inner surface of the plate;
$\Sigma_i, \;\; i = \overline{1, 4} -$ the lateral faces of the plate;
$e_{kl}(\overline{u}) = \dfrac{1}{2}\Big(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\Big) -$ the small deformations tensor;
$\overline{u}^0 -$ the main term in the asymptotic expression of the displacement vector given by the homogenization method [11];
$\varepsilon \in (0, 1] -$ a parameter depending on the material(for instance $\varepsilon = 1$ for black);
$\partial\Omega = \Sigma^+ \bigcup \Sigma^- \bigcup \bigcup_{j=1}^{4} \Sigma_j -$ the boundary of Ω;
$\overline{n} -$ exterior normal to surface $\partial\Omega$;

The governing equations of the dynamical problem for thermoelastic thin porous plates are obtained using the homogenization method [11]; they represent o genelization of the above

$$\begin{cases} \dfrac{\partial}{\partial x_j}\left[a^H_{ijkl}\cdot e_{kl}-\beta\cdot\theta^0\cdot\delta_{ij}\right]=\rho_0\cdot\dfrac{\partial^2 u^0_i}{\partial t^2}, \\ \rho_0\cdot c_e\cdot\dfrac{\partial\theta^0}{\partial t}=\dfrac{\partial}{\partial x_j}\left(k^H_{ij}\cdot\dfrac{\partial\theta^0}{\partial x_i}\right) \end{cases} \tag{2.3}$$

in which the elasticity tensor $\left(a^H_{ijkl}\right)^3_{i,j,k,l=1}$ given by the homogenization method has the expression $a^H_{ijkl}=\dfrac{1}{|P|}\cdot\displaystyle\int_P a_{ijmh}\left[\delta_{mk}\delta_{hl}+e_{mhy}(w^{kl})\right]dy$ and the conduction-convection tensor also given by the homogenization method $\left(k^H_{ij}\right)^3_{i,j=1}$ is defined by $k^H_{ij}=k\cdot\left(\delta_{ij}+\dfrac{1}{|P|}\cdot\displaystyle\int_P\dfrac{\partial\beta_j}{\partial y_i}dy\right)$. Following the homogenization theory concepts from [11], we assume that P is a parallelepipedic cell of dimension $\epsilon_0<<1$.

Let us define the Hilbert space $\widetilde{H}^1_{per}(P)$(with respect to the norm from H^1),

$$\widetilde{H}^1_{per}(P)=\{v\in H^1(P): v \text{ is a } \; P- \text{ periodic function}, \; \frac{1}{|P|}\cdot\int_P v dy=0\}.$$

The vector fields $(w^{kl})^3_{k,l=1}$, $(\beta_j)^3_{j=1}$ are solutions for the following variational problems

$$\begin{cases} w^{kl}\in\widetilde{H}^1_{per}(P) \\ \displaystyle\int_P a_{ijmh}\cdot e_{mhy}(w^{kl})\cdot e_{ijy}(\overline{v})dy=\int_P\frac{\partial a_{ijkl}}{\partial y_i}\cdot v_j dy, \quad \forall\overline{v}\in\widetilde{H}^1_{per}(P) \end{cases} \tag{2.4}$$

respectively

$$\begin{cases} \beta_k\in\widetilde{H}^1_{per}(P) \\ \displaystyle\int_P\frac{\partial\beta_k}{\partial y_i}\cdot\frac{\partial v}{\partial y_i}dy=\int_P\frac{\partial v}{\partial y_k}dy, \quad \forall v\in\widetilde{H}^1_{per}(P). \end{cases} \tag{2.5}$$

The problem (2.3) with the corresponding boundary conditions

$$\begin{cases} \sum_{j=1}^3\sigma^H_{ij}\cdot n_j=0, \quad (i=\overline{1,3}), \quad \text{on } \partial\Omega \\ \sum_{j=1}^3 n_j\cdot k^H_{ij}\cdot\dfrac{\partial\theta^0}{\partial x_i}+\varepsilon\cdot\sigma\cdot({\theta^0}^4-{S^*}^4)=0, \text{ on } \Sigma^+: x_3=\dfrac{h_0}{2} \\ \sum_{j=1}^3 n_j\cdot k^H_{ij}\cdot\dfrac{\partial\theta^0}{\partial x_i}+c_t\cdot(\theta^0-S^*)=0, \text{ on } \displaystyle\bigcup_{j=1}^4\Sigma_j; \\ \theta^0=S^*, \quad \text{on } \Sigma^-: x_3=-\dfrac{h_0}{2} \end{cases} \tag{2.6}$$

and the following initial conditions [4]:

$$\begin{cases} u^0_i(x,0)=\widetilde{u}_i, \; i=\overline{1,3}, \; x\in\Omega \\ \theta^0(x,0)=\widetilde{\theta}, \; x\in\Omega \end{cases} \tag{2.7}$$

is an initial boundary value problem.

A variational formulation for the dynamical problem (2.3) has been written in $\widetilde{H}^1_{per}(P)$ space. Following [1] the functional defining the variational formulation is assumed to be a positive definite quadratic form.

3. THE UNIQUENESS OF THE SOLUTION

In order to formulate the result based on the energy partition, it is useful to introduce the following functions depending on t, $t \in \mathcal{T}$

$$K(t) = \frac{1}{2}\int_\Omega \rho_0 \cdot \dot{u}_i^0(x,t) \cdot \dot{u}_i^0(x,t)dx,\ W(t) = \frac{1}{2}\int_\Omega [\sigma_{ij}^H \cdot e_{ij}(\overline{u}^0) + {\theta^0}^2(x,t)]dx. \tag{3.8}$$

$$P(s) = \varepsilon \cdot \sigma \int_{\Sigma^+} \theta^0 \cdot (S^{*4} - {\theta^0}^4)da. \tag{3.9}$$

Lemma 3.1. *For each solution $(\overline{u}^0, \theta^0)$ of the initial-boundary value problem (2.3)-(2.6)-(2.7) the functions W and K from (3.8) satisfy the identity*

$$W(t) + K(t) + \int_0^t \int_\Omega \frac{1}{\rho_0 \cdot c_e} \cdot k_{ij}^H \cdot \frac{\partial \theta^0}{\partial x_i} \cdot \frac{\partial \theta^0}{\partial x_j} dxds - \int_0^t P(s)ds = W(0) + K(0). \tag{3.10}$$

Proof. The proof of the Lemma is based on the identity

$$\frac{1}{2} \cdot \frac{d}{dt}[\sigma_{ij}^H \cdot e_{ij}(\overline{u}^0) + {\theta^0}^2(x,t)] = \sigma_{ij}^H \cdot \dot{e}_{ij}(\overline{u}^0) + \theta^0(x,t) \cdot \dot{\theta}^0(x,t).$$

The above relation is integrated over Ω and taking into account all the equations that define the dynamical problem (2.3), the definition of the e_{ij} tensor and the divergence theorem, we get

$$\frac{d}{dt}[W(t) + K(t)] = \varepsilon \cdot \sigma \int_{\Sigma^+} \theta^0 \cdot (S^{*4} - {\theta^0}^4)da - \int_\Omega \frac{1}{\rho_0 \cdot c_e} \cdot k_{ij}^H \cdot \frac{\partial \theta^0}{\partial x_i} \cdot \frac{\partial \theta^0}{\partial x_j} dx.$$

Integrating the above relation over $[0,t]$ the result of the theorem is proved. □

Theorem 3.1. *For $r, s \in \mathcal{T}$ let us introduce the new function $Y(r,s)$ defined by*

$$Y(r,s) := \varepsilon \cdot \sigma \int_{\Sigma^+} \theta^0(x,r) \cdot (S^{*4} - {\theta^0}^4(x,s))dx.$$

Then $W(t) - K(t) = \frac{1}{2}\int_0^t \frac{1}{\rho_0 \cdot c_e} \cdot [Y(t+s, t-s) - Y(t-s, t+s)]ds +$

$$+\frac{1}{2}\int_\Omega [\sigma_{ij}^H(2t) \cdot e_{ij}(0) + \theta^0(2t) \cdot \theta^0(0)]dx - \frac{1}{2}\int_\Omega \rho_0 \cdot \dot{u}_i(2t) \cdot \dot{u}_i(0)dx. \tag{3.11}$$

Proof. For arbitrary $t, s \in \mathcal{T}$ we can introduce the following notation

$$E(t,s) = \sigma_{ij}^H(t+s) \cdot e_{ij}(t-s) + \theta^0(t+s) \cdot \theta^0(t-s)$$

in which, in order to simplify the symbolic writing, the argument x was eliminated. The expressions of the σ_{ij}^H and e_{ij} tensors, lead to

$$\begin{aligned}
&\frac{\partial}{\partial s}E(t,s) = \dot{\sigma}_{ij}^H(t+s)\cdot e_{ij}(t-s) - \sigma_{ij}^H(t+s)\cdot \dot{e}_{ij}(t-s) + \dot{\theta}^0(t+s)\cdot\theta^0(t-s) - \\
&-\theta^0(t+s)\cdot\dot{\theta}^0(t-s) = a_{ijkl}^H\cdot\dot{e}_{kl}(t+s)\cdot e_{ij}(t-s) - \sigma_{ij}^H(t+s)\cdot\dot{e}_{ij}(t-s) + \\
&+\dot{\theta}^0(t+s)\cdot\theta^0(t-s) - \theta^0(t+s)\cdot\dot{\theta}^0(t-s) = [\sigma_{kl}^H(t-s)\cdot\dot{e}_{kl}(t+s) + \dot{\theta}^0(t+s)\cdot \\
&\cdot\theta^0(t-s)] - -[\sigma_{ij}^H(t+s)\cdot\dot{e}_{ij}(t-s) + \dot{\theta}^0(t-s)\cdot\theta^0(t+s)].
\end{aligned} \tag{3.12}$$

Integrating over Ω and taking into account the boundary conditions (2.6), we get

$$\int_\Omega \frac{\partial}{\partial s}E(t,s)dx = \frac{\partial}{\partial s}\int_\Omega \rho_0\cdot\dot{u}_i(t+s)\cdot\dot{u}_i(t-s)dx + \frac{1}{\rho_0\cdot c_e}\cdot[Y(t-s,t+s) - Y(t+s,t-s)].$$

Another integration step over $[0,t]$ of the above expression give us the result of the theorem □

Theorem 3.2. *The conduction-convection tensor (k_{ij}^H) is a positive definite tensor.*

Proof. Each component β_j, $j = 1,2,3$ from the definition of the conduction-convection tensor is the solution of the following equation

$$\begin{cases}
\triangle_{\overline{y}}\beta_j = 0, \quad \text{in } P \\
\overline{n}\cdot\nabla_{\overline{y}}\beta_j = -n_j, \quad \text{on } \partial P \\
\beta_j \;\; P - periodic \\
<\beta_j> = 0, \text{with} <\beta_j> = \dfrac{1}{|P|}\cdot\displaystyle\int_P \beta_j(y)dy.
\end{cases} \tag{3.13}$$

This imply $\displaystyle\int_P \frac{\partial\beta_j}{\partial y_i}dy = -\int_P \nabla_{\overline{y}}\beta_j\cdot\nabla_{\overline{y}}\beta_i dy$ so the conduction tensor k_{ij}^H is a symmetric operator, i.e. $k_{ij}^H = k_{ji}^H = k\cdot\left(\delta_{ij} - \dfrac{1}{|P|}\cdot\displaystyle\int_P \nabla_{\overline{y}}\beta_j\cdot\nabla_{\overline{y}}\beta_i dy\right)$.
In addition, let us define $B_j = -\beta_j - y_j$ in P. The conduction-convection tensor can be written $k_{ij}^H = k_{ji}^H = k\cdot <\nabla_{\overline{y}}B_i\cdot\nabla_{\overline{y}}B_j>$. With the representation we have $\sum_{i=1}^3\sum_{j=1}^3 k_{ij}^H\xi_i\xi_j \geq 0$ which establish the positivity of the conduction-convection tensor. □

Theorem 3.3. *Let $\rho_0 > 0$. Then the initial-boundary value problem (2.3) - (2.6) - (2.7) has a unique solution.*

Proof. Assume that two solution of the the initial boundary value problem (2.3) - (2.6) - (2.7) exists and denote these solutions with $\{u_i^1,\theta_1\}$ and $\{u_i^2,\theta_2\}$ respectively. Then, the difference between the two solutions, i.e. $\{u_i,\theta\}$, satisfies the equation (2.3), but with homogeneous initial and boundary conditions. Let us assume that for a fixed value of $t>0$ we have $\theta_1\geq\theta_2$. Substracting the expressions from (3.10)-(3.11) and evaluating the result for the difference of the two solutions, (u_i,θ), the equation reduces to

$$2K(t) + \int_0^t\int_\Omega \frac{1}{\rho_0\cdot c_e}\cdot k_{ij}^H\cdot\frac{\partial\theta}{\partial x_i}\cdot\frac{\partial\theta}{\partial x_j}dxds + \frac{1}{2}\int_0^t\int_{\Sigma^+}\frac{\varepsilon\cdot\sigma}{\rho_0\cdot c_e}\cdot f(r)\cdot g(r)dadr = 0. \tag{3.14}$$

Here $f(r) = \theta_1(r) - \theta_2(r)$ and $g(r) = [\theta_1(r) + \theta_2(r)] \cdot [\theta_1^2(r) + \theta_2^2(r)] \cdot [(\theta_1(2t-r) - \theta_1(r)) - (\theta_2(2t-r) - \theta_2(r))] - [{\theta_1}^4(2t-r) + {\theta_1}^4(r) - {\theta_2}^4(2t-r) - {\theta_2}^4(r)]$. In these conditions, there exists a value $\xi \in (0, t)$ such that

$$2K(t) + \int_0^t \int_\Omega \frac{1}{\rho_0 \cdot c_e} \cdot k_{ij}^H \cdot \frac{\partial \theta}{\partial x_i} \cdot \frac{\partial \theta}{\partial x_j} dx ds + \frac{1}{2} \int_{\Sigma^+} g(\xi) \int_0^t \frac{\varepsilon \cdot \sigma}{\rho_0 \cdot c_e} \cdot f(r) dr da = 0, \tag{3.15}$$

and, for $\rho_0 > 0$ and (k_{ij}^H) - a positive definite operator, we obtain $K(t) = 0$. We can also write $\int_0^t \int_\Omega \frac{1}{\rho_0 \cdot c_e} \cdot k_{ij}^H \cdot \frac{\partial \theta}{\partial x_i} \cdot \frac{\partial \theta}{\partial x_j} dx ds = 0$. This lead us to $\dot{u}_i = 0$, which imply that $u_i = 0$, on $\overline{\Omega} \times \mathcal{T}$. Similarly, we can write the relation for (u_i, θ) and obtain $W(t) = 0$, $t \in \mathcal{T}$ and $\theta = 0$, on $\overline{\Omega} \times \mathcal{T}$. This proves that the solution of the initial-boundary value problem (2.3) - (2.6) - (2.7) is unique. $\square$

4. Numerical investigations and conclusions

The physical domain characterizing the thin porous plates (a magnesium plate in our case) is given by $[0, L] \times [0, l] \times [-h_0/2, h_0/2]$, $L = 400mm$, $l = 250mm$, $h_0 = 8mm$ and with $S^* = 800K$. A complete characterization of the dynamical problem is given here by the isovalues for the vertical displacement fields. $displacement field : iteration 2$

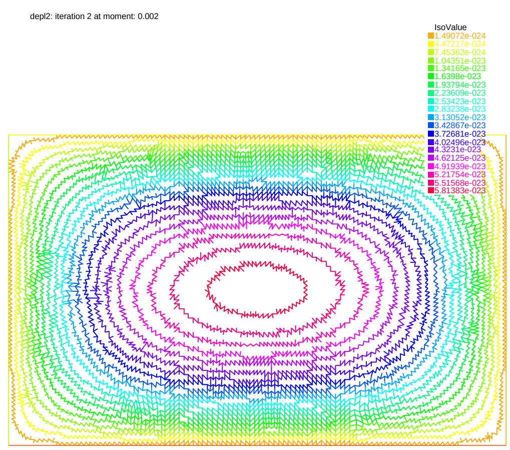

Figure 1. Numerical isovalues for the absolute temperature and the displacement fields in the case of a magnesium porous plates

The curves represented in Figure 1 are the isotherms curves connecting the points with the same displacement field. They are obtained using the Freefm++ soft based on the finite element method. In our case, even for the deformation process, the medium is a periodic one, so the porosity is considered as a constant value. Due to this periodicity propriety the isovalues are similar for an initial or a deformed domain. A natural conclusion is pointed out numerically: an increase of the temperature field until it reaches its maximum leads to an increase of the displacement field.

Another important aspect can be pointed here: we obtain a thermal equilibrium state which imply that the solution of the dynamical problem is stable in time. For the numerical study, the problem is reduced to the 2-dimensional case, using the micropolar theory of Eringen [5]. As the absolute temperature inside the plate is growing, the plate is deforming more and more until it reaches the thermal equilibrium state.

Acknowledgement. The first author is grateful to the CNCSIS grant AT 60/2007 and CNCSIS grant PN II 1081/2008 for the support during writing this paper.

REFERENCES

[1] Bârsan, M., *Deformarea placilor elastice poroase*, Ed. Matrix, 2007

[2] Chudinovich, I., Constanda, C. and Colin Venegas, J., *Solvability of initial-boundary value problems for bending of thermoelastic plates with mixed boundary conditions*, J. Math. Anal.Appl. **371** (2005), 357-376

[3] Ene, R. and Blaga, A.M., *Thermal radiation transport in a thin porous plate*, Proceedings of the International Conference on Theory and Applications of Mathematics and Informatics, ICTAMI 2007, Alba Iulia, Romania, pg. 105-113

[4] Ene, R. and Dragomirescu, I., *On the deformation of a thin thermoelastic plate*, Annals of Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity, **6** (2008), 45-55

[5] Eringen, A.C., *Theory of micropolar plates*, ZAMP, **18** (1967), 12-30

[6] Green, A.E. and Lindsay, K.A., *Thermoelasticity*, J of Elasticity, **2** (1972), 1-7

[7] Kosmodamianskii, A.S., *Flexure of anisotropic plates with curvilinear holes (survey)*, Int. Applied Mech., **17** (2) (1981), 103-109

[8] Lord, H.W. and Shulman, Y., *A generalized dynamical theory of thermoelasticity*, J of Mech. Phys. Solids, **15** (1967), 299-309

[9] Mitrinović, D.S., *Analytic inequalities*, Springer Verlag, Berlin-Heidelberg-New York, (1970)

[10] Podio-Guidugli, P., *An exact derivation of the thin plate equation*, J. Elasticity, **22** (1989), 121-133

[11] Sanchez-Hubert, J. and Sanchez-Palencia, E., *Introduction aux méthodes asymptotiques et à l'homogénéisation*, MASSON, (1992)

[12] Sugano, Y., *On a stress function method of a thermoelastic problem expressed in cylindrical coordinates in a multiply-connected region exibiting temperature-dependent material properties*, Ingenieur-Archiv **54** (1984), 301-308

UNIVERSITY "POLITEHNICA" OF TIMISOARA
DEPARTMENT OF MATHEMATICS
PIAŢA VICTORIEI NO. 2
300006 TIMIŞOARA, ROMANIA
E-mail address: eneremus@gmail.com
E-mail address: i.dragomirescu@gmail.com