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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} anniversary

A multivalued Perov-type theorem in generalized metric spaces

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ABSTRACT. In this paper we recall the concept of multivalued weakly Picard operator. Then we present a fixed points result for multivalued contractive type operators with respect to a w-distance.

1. INTRODUCTION

The concept of multivalued weakly Picard operator (briefly MWP operator) was introduced in close connection with the successive approximation method and the data dependence phenomenon for the fixed point set of multivalued operators on complete metric space, by I. A. Rus, A. Petruşel and A. Sântămărian, see [9]. In 1966 A. I. Perov introduced the concept of generalized metric space and obtained a generalization of the Banach principle for contractive operators on spaces endowed with vector-valued metrics, see [7].

In [8] the theory of multivalued weakly Picard operators in L-spaces is presented. In 1976, Caristi [1] proved a fixed point theorem in the framework of complete metric spaces which is a generalization of the Banach contraction principle. Another interesting results in different spaces was obtain in [11], [13], [3]. Later, in 1996, O. Kada, T. Suzuki and W. Takahashi [4] introduced the concept of w-distance on a metric space and by using this new concept they obtained a generalization of Caristi's fixed point theorem.

The purpose of this paper is to define the notion of generalized w-distance in a generalized metric space and to present fixed point results for multivalued weakly Picard operators in generalized complete metric spaces endowed with a generalized w-distance.

2. PRELIMINARIES

Let (X, d) be a complete metric space. We will use the following notations:
$P(X)$ - *the set of all nonempty subsets of X*;
$\mathcal{P}(X) = P(X) \bigcup \emptyset$
$P_{cl}(X)$ - *the set of all nonempty closed subsets of X*;
$P_b(X)$ - *the set of all nonempty bounded subsets of X*;
$P_{b,cl}(X)$ - *the set of all nonempty bounded and closed, subsets of X*;

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For two subsets $A, B \in P_b(X)$ we recall the following functionals.

$D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, D(Z,Y) = inf\{d(x,y) : x \in Z \ , y \in Y\}$ - *the gap functional*.

$\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, \delta(A,B) := sup\{d(a,b) | x \in A, b \in B\}$ - *the diameter functional*;

$\rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, \rho(A,B) := sup\{D(a,B) | a \in A\}$ - *the excess functional*;

$H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, H(A,B) := max\{\sup_{a\in A} \inf_{b\in B} d(a,b), \sup_{b\in B} \inf_{a\in A} d(a,b)\}$ - *the Pompeiu-Hausdorff functional*;

First we define the concept of **L-space**.

Definition 2.1. Let X be a nonempty set and $s(X) := \{(x_n)_{n\in\mathbb{N}} | x_n \in X, n \in \mathbb{N}\}$. Let $c(X) \subset s(X)$ a subset of $s(X)$ and $Lim : c(X) \to X$ an operator. By definition the triple $(X, c(X), Lim)$ is called an **L-space** if the following conditions are satisfied:

(i) If $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.

(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences, $(x_{n_i})_{i\in\mathbb{N}}$, of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.

By the definition an element of $c(X)$ is convergent and $x := Lim(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence and we can write $x_n \to x$ as $n \to \infty$.

We will denote an L-space by $(X, \to)$.

Let us give some examples of L-spaces, see [8].

Example 2.1. (L-structures on Banach spaces). Let X be a Banach space. We denote by $\to$ the strong convergence in X and by $\rightharpoonup$ the weak convergence in X. Then $(X, \to)$, $(X, \rightharpoonup)$ are L-spaces.

Example 2.2. (L-structures on function spaces). Let X and Y be two metric spaces. Let $\mathbb{M}(X,Y)$ the set of all operators from X to Y. We denote by $\stackrel{p}{\to}$ the pointwise convergence on $\mathbb{M}(X,Y)$, by $\stackrel{unif}{\to}$ the uniform convergence and by $\stackrel{cont}{\to}$ the convergence with continuity. Then $(\mathbb{M}(X,Y), \stackrel{p}{\to})$, $(\mathbb{M}(X,Y), \stackrel{unif}{\to})$ and $(\mathbb{M}(X,Y), \stackrel{cont}{\to})$ are L-spaces.

Definition 2.2. *Let $(X, \to)$ be an L-space. Then $T : X \to P(X)$ is a* **multivalued weakly Picard operator***(briefly MWP operator)if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:*

(i)$x_0 = x$, $x_1 = y$;

(ii)$x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$;

(iii)the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

Let us give some examples of MWP operators, see [8],[9].

Example 2.3. Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a Reich type multivalued operator, i.e. there exists $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $\alpha + \beta + \gamma < 1$ such that

$$H(T(x), T(y)) \leq \alpha d(x,y) + \beta D(x, T(x)) + \gamma D(y, T(y)),$$

for all $x, y \in X$. Then T is a MWP operator.

Example 2.4. Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a closed multifunction for which there exists $\alpha, \beta \in \mathbb{R}_+$ with $\alpha + \beta < 1$ such that

$H(T(x),T(y)) \leq \alpha d(x,y) + \beta D(y,T(y))$, for every $x \in X$ and every $y \in T(x)$. Then T is a MWP operator.

Example 2.5. *Let (X,d) be a complete metric space and $T_1, T_2 : X \to P_{cl}(X)$ for which there exists $\alpha \in]0, \frac{1}{2}[$ such that*

$$H(T_1(x), T_2(y)) \leq \alpha[D(x, T_1(x)) + D(y, T_2(y))],$$

for each $x, y \in X$. Then T_1 and T_2 are a MWP operators.

For the proof of the main result we need the following theorem, see [10].

Theorem 2.1. *Let $A \in M_{n,n}(\mathbb{R})$. The following statements are equivalent:*
(i)$A^k \to 0$ as $k \to \infty$.
(ii)The eigenvalues of A are in the open unit disc.
(iii)The matrix $I - A$ is an invertible matrix and

$$(I - A)^{-1} = I + A + ... + A^n + ...,$$

where I is the unit matrix.

The concept of w-distance was introduced in [4] as follows:

A mapping $w : X \times X \to \mathbb{R}_+$ is said to be w-distance on the metric space (X,d) if the following axioms are satisfied:

(w_1) For any $x, y, z \in X$ the inequality $w(x,z) \leq w(x,y) + w(y,z)$ holds;

(w_2) For every $x \in X$, the map $w(x,.) : X \to \mathbb{R}_+$ is lsc;

(w_3) For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $w(z,x) \leq \delta$ and $w(z,y) \leq \delta$, then $d(x,y) \leq \varepsilon$.

Examples of non trivial w-distances can be found in [4].

A crucial result in order to obtain fixed point theorems by using a w-distance is the following:

Lemma 2.1 (see [4]). *Let (X,d) be a metric space, and let w be a w-distance on X. Let (x_n) and (y_n) be two sequences in X, let (α_n), (β_n) be sequences in $[0, +\infty[$ converging to zero and let $x, y, z \in X$. Then the following hold:*

(1) If $w(x_n, y) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$.
(2) If $w(x_n, y_n) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z.
(3) If $w(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.
(4) If $w(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.

The above lemma is used to prove the following generalization of Caristi's fixed point theorem [1]:

Theorem 2.2. *Let (X,d) be a complete metric space, let w be a w-distance on X and let $T : X \to X$ be a mapping such that there exists $r \in [0,1)$ satisfying*

$$w(T(x), T^2(x)) \leq rw(x, T(x))$$

for every $x \in X$ and that

$$\inf\{w(x,y) + w(x,T(x)) : \ x \in X\} > 0,$$

for every $y \in X$ with $y \neq T(y)$. Then T has a fixed point.

3. Generalized w-distance

Recall first the concept of generalized metric in Perov's sense, see [7] and [10].

We consider in $\mathbb{R}^m$ the natural order relation, i.e., if $x, y \in \mathbb{R}^m$, $x = (x_1, x_2, ...x_m)$, $y = (y_1, y_2, ...y_m)$ then $x \leq y$ if and only if $x_i \leq y_i$, for $i = \overline{1, m}$.

Definition 3.1. Let X be a set. A mapping $\widetilde{d} : X \times X \to \mathbb{R}^m_+$ is called a generalized metric if there are accomplished the following conditions:

(i) $\widetilde{d}(x, y) \geq 0$ for every $x, y \in X$; in particular if $\widetilde{d}(x, y) = 0$ then $x = y$;

(ii) $\widetilde{d}(x, y) = \widetilde{d}(y, x)$ for every $x, y \in X$;

(iii) $\widetilde{d}(x, y) \leq \widetilde{d}(x, z) + \widetilde{d}(z, y)$ for every $x, y, z \in X$.

Definition 3.2. A set X together with a generalized metric $\widetilde{d}$ defined above forms a generalized metric space.

Remark 3.1. The notions of convergent sequence, fundamental sequence, generalized complete metric space, generalized metric induced by a generalized norm are defined in the same way than for the usual metric spaces.

We will introduce now the concept of generalized w-distance.

Definition 3.3. Let (X, d) a generalized metric space. The mapping $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ defined by $\widetilde{w}(x, y) = (v_1(x, y), v_2(x, y), ..., v_m(x, y))$ is called generalized w-distance if the following conditions hold:

(w_1) $\widetilde{w}(x, y) \leq \widetilde{w}(x, z) + \widetilde{w}(z, y)$, for every $x, y, z \in X$;

(w_2) $v_i : X \times X \to \mathbb{R}_+$ is lower semicontinuous for $i = \overline{1, m}$;

(w_3) For any $\varepsilon := (\varepsilon_1, \varepsilon_2, ..., \varepsilon_m) \in \mathbb{R}^*_+$ (where $m \in \mathbb{N}^*$), there exists $\delta := (\delta_1, \delta_2, ..., \delta_m) \in \mathbb{R}^*_+$ such that $\widetilde{w}(z, x) \leq \delta$ and $\widetilde{w}(z, y) \leq \delta$ implies $\widetilde{d}(x, y) \leq \varepsilon$.

Let us translate the crucial lemma for w-distance in the terms of generalized w-distance.

Lemma 3.1. *Let $(X, \widetilde{d})$ be a generalized metric space, and let $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w-distance on X. Let (x_n) and (y_n) be two sequences in X, let $\alpha_n = (\alpha_n^{(1)}, \alpha_n^{(2)}, ..., \alpha_n^{(m)}) \in \mathbb{R}_+$ and $\beta_n = (\beta_n^{(1)}, \beta_n^{(2)}, ..., \beta_n^{(m)}) \in \mathbb{R}_+$ be two sequences such that $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ converge to zero for each $i = \overline{1, m}$. Let $x, y, z \in X$. Then the following assertions hold:*

1. *If $\widetilde{w}(x_n, y) \leq \alpha_n$ and $\widetilde{w}(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$.*
2. *If $\widetilde{w}(x_n, y_n) \leq \alpha_n$ and $\widetilde{w}(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z.*
3. *If $\widetilde{w}(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.*
4. *If $\widetilde{w}(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.*

Let us present some examples of generalized w-distance.

Example 3.1. Let $(X, \widetilde{d})$ be a generalized metric space, and let $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w-distance on X. Then $\widetilde{w} = \widetilde{d}$ is a generalized w-distance on X.

Proof. The condition (w_1) are accomplish for all $d_i(x, y)$ with $i = \overline{1, m}$, for $m \in \mathbb{N}$, from the positions of the generalized metric $\widetilde{d}(x, y)$ by the triangle inequality of the definition of a usual metric. Thus $\widetilde{d}(x,y) \leq \widetilde{d}(x,z) + \widetilde{d}(z,y)$, for every $x, y, z \in X$.

Let $\{y_p\}_{p\in\mathbb{N}}$ be a convergent sequence in X and let $y \in X$ be the limit of this sequence.

$$\widetilde{d}(x,y) = (d_1(x,y), d_2(x,y), ..., d_n(x,y)) \leq$$
$$\leq (\lim_{p\to\infty} d_1(x,y_p), \lim_{p\to\infty} d_2(x,y_p), ..., \lim_{p\to\infty} d_m(x,y_p)) \leq \lim_{p\to\infty} \widetilde{d}(x,y_p)$$

Then the application $\widetilde{d}(x,\cdot) : X \to \mathbb{R}^m_+$ is l.s.c. and is accomplished the second condition (w_2) of the definition of generalized w-distance.

Let $\varepsilon := (\varepsilon_1, \varepsilon_2, ..., \varepsilon_m) \in \mathbb{R}^*_+$ be given for $m \in \mathbb{N}$ and put $\delta := (\delta_1, \delta_2, ..., \delta_m) \in \mathbb{R}^*_+$ such that $\delta = \frac{\varepsilon}{2}$.Then we have true $\widetilde{d}(z,x) \leq \delta$ and $\widetilde{d}(z,y) \leq \delta$. Then by triangle inequality and the symmetry of the usual metric result

$$\widetilde{d}(x,y) \leq \widetilde{d}(x,z) + \widetilde{d}(z,y) \leq \delta + \delta = \varepsilon$$

In this case $\widetilde{d}(x,y)$ accomplish the condition (w_3) of the definition of generalized w-distance. □

Example 3.2. Let $w_1, ..., w_m : X \times X \to \mathbb{R}_+$ be w-distances. Then $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ defined by $\widetilde{w}(x,y) = (w_1(x,y), w_2(x,y), ..., w_m(x,y))$ is a generalized w-distance.

Proof. It is easy to remark that the conditions (w_1) and (w_3) from the definition of generalized w-distance are satisfied of each w-distance w_i, for $i = \overline{1,m}$. The second condition (w_2) from the definition of generalized w-distance result by the lower semicontinuity of the w-distances $w_1, w_2, ...w_m$. □

4. Main results

The first result of our paper is a generalization of Theorem 1 from [12] in the terms of Perov-type theorems for generalized metric spaces endowed with a generalized w-distance.

Theorem 4.1. *Let $(X, \widetilde{d})$ be a complete generalized metric space and $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w-distance on X. Let $T : X \to P_{cl}(X)$ be a multivalued operator. Suppose that there exists $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ with $A^n \to 0$ as $n \to \infty$ such that for each $x, y \in X$ and each $u \in T(x)$ there exists $v \in T(y)$ with the following property*

$$\widetilde{w}(u,v) \leq A\widetilde{w}(x,y)$$

Then there exists $x^* \in X$ such that $x^* \in T(x^*)$ and, moreover, $w(x^*,x^*) = 0$.

Proof. Let $x_0 \in X$ and $x_1 \in T(x_0)$. Then exists $x_2 \in T(x_1)$ for which we have $\widetilde{w}(x_1,x_2) \leq A\widetilde{w}(x_0,x_1)$. Thus we can define the sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that $x_{n+1} \in T(x_n)$ and $\widetilde{w}(x_n,x_{n+1}) \leq A^n\widetilde{w}(x_{n-1},x_n)$ for every $n \in \mathbb{N}$.

Then we have, for any $n \in \mathbb{N}$,

$$\widetilde{w}(x_n,x_{n+1}) \leq A\widetilde{w}(x_{n-1},x_n) \leq ... \leq A^n\widetilde{w}(x_0,x_1).$$

Hence, for any $m, n \in \mathbb{N}$ with $m > n$, and using Theorem 2.8. result:

$$\widetilde{w}(x_n,x_{n+m}) \leq \widetilde{w}(x_n,x_{n+1}) + \widetilde{w}(x_{n+1},x_{n+2}) + ... + \widetilde{w}(x_{n+m-1},x_{n+m}) \leq$$

$$\leq A^n\widetilde{w}(x_0,x_1)+A^{n+1}\widetilde{w}(x_0,x_1)+...+A^{n+m-1}\widetilde{w}(x_0,x_1)\leq$$
$$\leq A^n(I-A)^{-1}\widetilde{w}(x_0,x_1).$$

From hypothesis we have that $A^n\to 0$ as $n\to\infty$. Using Lemma 3.5.(3) we have that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Thus $x_0=x$, $x_1=y$ and $x_{n+1}\in T(x_n)$.

Since $(X,\widetilde{d})$ is a complete space then the sequence $(x_n)_{n\in\mathbb{N}}$ it is a convergent sequence. Let $z\in X$ be the limit of the sequence $(x_n)_{n\in\mathbb{N}}$.

For $m>n$ we have the inequality:

$$\widetilde{w}(x_n,x_m)\leq\widetilde{w}(x_n,x_{n+1})+\widetilde{w}(x_{n+1},x_{n+2})+...+\widetilde{w}(x_{m-1},x_m)\leq$$
$$\leq A^n\widetilde{w}(x_0,x_1)+A^{n+1}\widetilde{w}(x_0,x_1)+...+A^{m-1}\widetilde{w}(x_0,x_1)\leq$$
$$\leq A^n(I-A)^{-1}\widetilde{w}(x_0,x_1).$$

By the Lemma 3.5.(3) we have that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $(X,\widetilde{w})$ is a complete metric space then there exists $x^*\in X$ such that $\lim\limits_{n\to\infty}x_n=x^*$.

Since for each $x_n\in X$ the operator $\widetilde{w}(x_n,.):X\to\mathbb{R}^m_+$ is l.s.c. for every $m\in\mathbb{N}$ we derive

$$\widetilde{w}(x_n,x^*)\leq\lim\inf_{m\to\infty}\widetilde{w}(x_n,x_m)\leq A^n(I-A)^{-1}\widetilde{w}(x_0,x_1).$$

Thus, for every $n\in\mathbb{N}$, $\widetilde{w}(x_n,x^*)\leq A^n(I-A)^{-1}\widetilde{w}(x_0,x_1)$.

For $x^*\in X$ and $x_n\in T(x_{n-1})$ there exists $u_n\in T(x^*)$ such that

$$\widetilde{w}(x_n,u_n)\leq A\widetilde{w}(x_{n-1},x^*)\leq...\leq A^n\widetilde{w}(x_0,x_1)$$

Therefore, we obtain that:
$\widetilde{w}(x_n,u_n)\leq A^n\widetilde{w}(x_0,x_1)$
$\widetilde{w}(x_n,x^*)\leq A^n(I-A)^{-1}\widetilde{w}(x_0,x_1)$

Then, by the Lemma 3.5.(2), we obtain that $u_n\xrightarrow{d}x^*$. As $u_n\in T(x^*)$ and using the closure of T result that $x^*\in T(x^*)$.

For $x^*\in X$ and $x^*\in T(x^*)$, using the hypothesis, there exists $z_1\in T(x^*)$ such that

$$\widetilde{w}(x^*,z_1)\leq A\widetilde{w}(x^*,x^*).$$

For $x^*,z_1\in X$ and $x^*\in T(x^*)$ there exists $z_2\in T(z_1)$ such that

$$w(x^*,z_2)\leq A\widetilde{w}(x^*,z_1).$$

By induction we get a sequence $(z_n)_{n\in\mathbb{N}}\in X$ such that

(i) $z_{n+1}\in T(z_n)$, for every $n\in\mathbb{N}$;

(ii) $\widetilde{w}(x^*,z_n)\leq A\widetilde{w}(x^*,z_{n-1})$, for every $n\in\mathbb{N}\setminus\{0\}$.

Therefore we have

$$\widetilde{w}(x^*,z_n)\leq A\widetilde{w}(x^*,z_{n-1})\leq A^2\widetilde{w}(x^*,z_{n-2})\leq\cdots\leq$$
$$\leq A^{n-1}\widetilde{w}(x^*,z_1)\leq A^n\widetilde{w}(x^*,x^*).$$

Thus $\widetilde{w}(x^*,z_n)\leq A^n\widetilde{w}(x^*,x^*)$.

When $n \to \infty$, $A^n\widetilde{w}(x^*,x^*)$ converge to 0. Thus, by the Lemma 3.5.(4) we obtain that $(z_n)_{n\in\mathbb{N}} \in X$ is a Cauchy sequence in $(X,\widetilde{d})$ and there exists $z^* \in X$ such that $z_n \overset{d}{\to} z^*$.

Using the lower semicontinuity of the components of the mapping $\widetilde{w}(x^*,\cdot)$, we have that

$$0 \leq \widetilde{w}(x^*,z^*) \leq \lim_{n\to\infty} \inf \widetilde{w}(x^*,z_n) \leq \lim_{n\to\infty} A^n\widetilde{w}(x^*,x^*) = 0.$$

Then $\widetilde{w}(x^*,z^*) = 0$.

So, by triangle inequality we have

$$\widetilde{w}(x_n,z^*) \leq \widetilde{w}(x_n,x^*) + \widetilde{w}(x^*,z^*) \leq A^n\widetilde{w}(x_0,x_1).$$

Since $A^n\widetilde{w}(x_0,x_1)$ converge to 0 when $n \to \infty$ we have

$$\widetilde{w}(x_n,z^*) \leq A^n\widetilde{w}(x_0,x_1)$$
$$\widetilde{w}(x_n,x^*) \leq A^n\widetilde{w}(x_0,x_1)$$

Using Lemma 3.5.(1) we have $z^* = x^*$, then $\widetilde{w}(x^*,x^*) = 0$. □

Remark 4.1. Notice that, in the conditions of the above theorem, T is a MWP operator.

The second result is a fixed point theorem for MWP operators in generalized metric space with respect to a generalized w-distance.

Theorem 4.2. *Let $(X,\widetilde{d})$ be a complete generalized metric space and $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w-distance on X. Let $T : X \to P(X)$ be a multivalued operator.*

Suppose that:

(i) there exists $A \in \mathcal{M}_{m,m}(\mathbb{R})$ with $A^n \to 0$ as $n \to \infty$ such that for each $x,y \in X$ and each $u \in T(x)$ there exists $v \in T(y)$ with the following property

$$\widetilde{w}(u,v) \leq A\widetilde{w}(x,y)$$

(ii) for every $x,y \in X$, with $y \notin T(y)$ we have that

$$\inf\{\widetilde{w}(x,y) + D_{\widetilde{w}}(x,T(x)) :\ x \in X\} > 0,$$

where $D_{\widetilde{w}}(x,T(x)) = \inf\{\widetilde{w}(x,y) :\ y \in T(x)\}$.

Then T is a MWP operator.

Proof. In the same way as in the proof of the Theorem 4.1. we construct a sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that

(i) $x_{n+1} \in T(x_n)$

(ii) $\widetilde{w}(x_n,x_{n+1}) \leq A\widetilde{w}(x_{n-1},x_n)$ for every $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$,

$$\widetilde{w}(x_n,x_{n+1}) \leq A\widetilde{w}(x_{n-1},x_n) \leq ... \leq A^n\widetilde{w}(x_0,x_1).$$

Hence, for any $m,n \in \mathbb{N}$ with $m > n$ and using Theorem 2.8. we have:

$$\widetilde{w}(x_n,x_m) \leq A^n(I-A)^{-1}\widetilde{w}(x_0,x_1).$$

From hypothesis (i) we have that $A^n \to 0$ as $n \to \infty$. Using Lemma 3.5.(3) we have that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Thus $x_0 = x$, $x_1 = y$ and $x_{n+1} \in T(x_n)$.

Since $(X, \widetilde{d})$ is a complete generalized metric space then the sequence $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence. Let $x^* \in X$ be the limit of the sequence $(x_n)_{n\in\mathbb{N}}$.

Assume that $x^* \notin T(x^*)$. Since for each $x_n \in X$ the operator $\widetilde{w}(x_n, .) : X \to \mathbb{R}^m_+$ is l.s.c. for every $m \in \mathbb{N}$, we derive

$$\widetilde{w}(x_n, z) \leq \liminf_{m\to\infty} \widetilde{w}(x_n, x_m) \leq A^n(I - A)^{-1}\widetilde{w}(x_0, x_1).$$

Therefore by hypothesis (ii) and by using the above inequality, we obtain

$$\begin{aligned} 0 \quad & < \inf\{\widetilde{w}(x, x^*) + D_{\widetilde{w}}(x, T(x)) : x \in X\} \\ & \leq \inf\{\widetilde{w}(x_n, x^*) + \widetilde{w}(x_n, x_{n+1}) : \ n \in \mathbb{N}\} \\ & \leq \inf\{2A^n(I - A)^{-1}\widetilde{w}(x_0, x_1) : n \in \mathbb{N}\} \\ & = \lim_{n\to\infty} 2A^n(I - A)^{-1}\widetilde{w}(x_0, x_1) = 0. \end{aligned}$$

Which is a contradiction. Thus we conclude that $x^* \in T(x^*)$.

Then T is a MWP operator. □

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