CREATIVE MATH. & INF. **Online version at** `http://creative-mathematics.ubm.ro/`
17 (2008), No. 3, 427 - 430 **Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X**

Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} *anniversary*

A Perov-type fixed point theorem in generalized ordered metric spaces

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ABSTRACT. In this paper we prove a Perov-type fixed point theorem in generalized ordered metric spaces.

1. INTRODUCTION

In 1966 Perov formulated a fixed point theorem which extends the well-known contraction mapping principle for the case when the metric d takes values in $\mathbb{R}_+^m$, that is, in the case when we have a generalized metric space.

The full statement of Perov's fixed point theorem, see [5] is the following:

Theorem P. *Let* (X, d) *be a generalized complete metric space* $(d(x, y) \in \mathbb{R}_+^m)$ *and* $f : X \to X$ *a mapping which satisfies the condition*

$$d(f(x), f(y)) \leq Ad(x, y),\ \forall\, x, y \in X \tag{1.1}$$

where $A \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$ *is a matrix convergent to zero.*
Then

(i) $F_f = \{x^*\}$, $F_f = \{x \in X | f(x) = x\}$;

(ii) *The sequence of successive approximations* $x_n = f^n(x_0)$ *is convergent and* $\lim\limits_{n\to\infty} x_n = x^*$, *for any* $x_0 \in X$;

(iii) *We have the estimation*

$$d(x_n, x^*) \leq A^n(I - A)^{-1} d(x_0, x_1)$$

In [4] A. C. M. Ran and M. C. B. Reurings generalized the Banach's fixed point theorem from usual metric spaces to ordered metric spaces.

In the present paper we give a Perov-type fixed point theorem in generalized ordered metric spaces. In this setting, the map assumed to be monotone satisfies a Lipschitz type condition with a matrix A. This condition is assumed to hold only on elements that are comparable with respect to the partial order. It is also assumed that f is continuous.We show that under such conditions a Perov-type fixed point theorem still hold.

Received: 10.09.2008. In revised form: 02.05.2009. Accepted: 22.05.2009.
2000 *Mathematics Subject Classification*. 47H10.
Key words and phrases. *Generalized ordered metric space, fixed points.*

2. MAIN RESULT

In this section we will prove the following fixed point theorem

Theorem 2.1. *Let X be a partially ordered set such that every pair $x, y \in X$ has a lower and an upper bound. Furthermore, let d be a metric on X such that (X, d) is a generalized complete metric space ($d(x, y) \in \mathbb{R}^m_+$). If the map $f : X \to X$ is continuous, monotone (i.e., increasing or decreasing) such that*

$1)$ *f satisfies a Lipschitz type condition with a matrix $A \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$*

$$d(f(x), f(y)) \le Ad(x, y),\ \forall\, x \ge y; \tag{2.1}$$

$2)$ *$A^n \to O$, $n \to \infty$;*

$3)$ *$\exists\, x_0 \in X$ such that $x_0 \le f(x_0)$ or $x_0 \ge f(x_0)$.*

Then

(i) *$F_f = \{x^*\}$;*

(ii) *The sequence of successive approximations $x_n = f^n(x)$ is convergent and $\lim\limits_{n\to\infty} f^n(x) = x^*$, $\forall\, x \in X$.*

Proof. Let $x_0 \in X$ be such that $x_0 \le f(x_0)$ or $x_0 \ge f(x_0)$. The monotonicity of f implies that $f^n(x_0) \le f^{n+1}(x_0)$ or $f^n(x_0) \ge f^{n+1}(x_0)$, for $n = 0, 1, 2, \ldots$

From $x_0 \le f(x_0)$ or $x_0 \ge f(x_0)$ by (2.1) we have

$$d(f(f(x_0)), f(x_0)) \le Ad(f(x_0), x_0). \tag{2.2}$$

We suppose that $d(f^n(x_0), f^{n-1}(x_0)) \le A^{n-1}d(f(x_0), x_0)$ and prove that $d(f^{n+1}(x_0), f^n(x_0)) \le A^n d(f(x_0), x_0)$.

From (2.1) we have

$$\begin{aligned}
&d(f(f^n(x_0)), f(f^{n-1}(x_0))) \le Ad(f^n(x_0), f^{n-1}(x_0))\\
&\le A \cdot A^{n-1}d(f(x_0), x_0) = A^n d(f(x_0), x_0).
\end{aligned}$$

In conclusion,

$$d(f^{n+1}(x_0), f^n(x_0)) \le A^n d(f(x_0), x_0), \forall\, n \in \mathbb{N}. \tag{2.3}$$

Now, we will prove that the sequence of successive approximations is Cauchy sequence

$$\begin{aligned}
&d(f^n(x_0), f^{n+p}(x_0)) \le d(f^n(x_0), f^{n+1}(x_0)) + d(f^{n+1}(x_0), f^{n+2}(x_0)) + \ldots\\
&+ d(f^{n+p-1}(x_0), f^{n+p}(x_0)) \overset{(2.3)}{\le} A^n d(x_0, f(x_0)) + A^{n+1}d(x_0, f(x_0)) + \ldots\\
&+ A^{n+p-1}d(x_0, f(x_0)) = (A^n + A^{n+1} + \cdots + A^{n+p-1})d(x_0, f(x_0))\\
&= A^n(I + A + \cdots + A^{p-1})d(x_0, f(x_0))\\
&\le A^n(I + A + \cdots + A^{p-1} + \ldots)d(x_0, f(x_0))\\
&= A^n(I - A)^{-1}d(x_0, f(x_0)),
\end{aligned}$$

$$d(f^n(x_0), f^{n+p}(x_0)) \le A^n(I - A)^{-1}d(x_0, f(x_0)) \tag{2.4}$$

which shows that the sequence $x_n = f^n(x_0)$ is fundamental sequence. The space (X, d) is complete, so (x_n) is convergent.

Let $x^* = \lim_{n\to\infty} x_n$. We have $x_{n+1} = f(x_n)$ and by letting $n \to \infty$ and f continuous we get

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x^*)$$

$x^* = f(x^*)$, thus $x^* \in F_f$.

It remains to show that x^* is the unique fixed point of f. We will show this by $\lim_{n\to\infty} f^n(x) = x^*$, for all $x \in X$.

For $x \leq x_0$ and $x \geq x_0$ it is obvious $f^n(x) \leq f^n(x_0)$ or $f^n(x) \geq f^n(x_0)$. We get

$$d(f^n(x), f^n(x_0)) \leq A^n d(x, x_0).$$

Because $A^n \to O$, $n \to \infty$ we have

$$\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f^n(x_0) = x^*.$$

Let $x \in X$ arbitrary and let x_1, respective x_2, be an upper bound and a lower bound of x and x_0. Then $x_1 \geq x \geq x_2$ imply

$$f^n(x_1) \geq f^n(x) \geq f^n(x_2) \quad \text{or} \quad f^n(x_1) \leq f^n(x) \leq f^n(x_2) \tag{2.5}$$

and $x_2 \leq x_0 \leq x_1$ imply $f^n(x_2) \leq f^n(x_0) \leq f^n(x_1)$ which yields

$$\lim_{n\to\infty} f^n(x_1) = \lim_{n\to\infty} f^n(x_2) = x^*. \tag{2.6}$$

From (2.5) and (2.6) we have $\lim_{n\to\infty} f^n(x) = x^*, \forall\, x \in X$. The proof is complete. □

Remark 2.1. Condition (2.1) is weaker than the condition (1.1) in Perov original fixed point theorem, where it is required that (1.1) is satisfied for all $x, y \in X$.

3. A Maia-Perov fixed point theorem

We give in this section a Perov-Maia type theorem in generalized ordered metric spaces, see [3], [2].

Theorem 3.1. (Perov-Maia) *Let X be a nonempty set, partially ordered, such that every pair $x, y \in X$ has a lower and an upper bound. Let d and ρ be two metrics on X and $f : X \to X$ a mapping. We suppose that*

(i) $\;d(x,y) \leq \rho(x,y), \quad \forall\, x \geq y$;

(ii) (X, d) *is a generalized ordered complete metric space;*

(iii) $f : (X,d) \to (X,d)$ *is continuous mapping;*

(iv) f *is a monotone mapping;*

(v) *there exists a matrix $A \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$ convergent to zero, such that*

$$\rho(f(x), f(y)) \leq A\rho(x,y), \quad \forall\, x \geq y;$$

(vi) $\exists\, x_0 \in X$ *such that* $x_0 \leq f(x_0)$ *or* $x_0 \geq f(x_0)$.

Then, $F_f = \{x^*\}$.

Proof. Let $x_0 \in X$ be such that $x_0 \leq f(x_0)$ or $x_0 \geq f(x_0)$. Using that f is monotone, we get $f^n(x_0) \leq f^{n+1}(x_0)$ or $f^n(x_0) \geq f^{n+1}(x_0)$. From (iv) and (v) we have

$$\rho(f^n(x_0), f^{n+1}(x_0)) \leq A^n \rho(x_0, f(x_0))$$

and using a similar argument like in the proof of Theorem 2.1 we get that (x_n), $x_n = f^n(x_0)$ is Cauchy sequence in (X, ρ).

From (i), (x_n) is Cauchy sequence in (X,d). Using (ii), (x_n) is convergent, $f:(X,d)\to(X,d)$ is continuous mapping, results $x^*\in F_f$. From (v) we have $F_f=\{x^*\}$. □

Remark 3.1. By Theorem 3.1, in the case $d\equiv\rho$, we obtain the Perov-type fixed point theorem, that is, Theorem 2.1.

The fixed point theorems obtained could be used in order to obtain applications to matrix equations, similarly to the ones given in the paper [4] for contraction mapping principle in ordered metric spaces.

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