CREATIVE MATH. & INF. **Online version at** `http://creative-mathematics.ubm.ro/`
17 (2008), No. 3, 431 - 438 **Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X**

Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} *anniversary*

The crossing number of $P_5^2 \times C_n$

MARIÁN KLEŠČ[1,2] AND DANIELA KRAVECOVÁ[2]

ABSTRACT. Patil and Krishnnamurthy established family of graphs for which power graphs have crossing number one. This is the only result concerning crossing numbers of power of some graphs. Let P_m^2 denote the power of the path P_m. We start to determine crossing numbers of a new infinite family of graphs, concretely for the Cartesian products $P_m^2 \times C_n$ where $m \geq 2$ and $n \geq 3$. The main result of the paper is that the crossing number of the graph $P_5^2 \times C_n$ is $4n$ for all $n \geq 3$.

1. INTRODUCTION

The $crossing\ number\ cr(G)$ of a simple graph G with vertex set V and edge set E is defined as the minimum number of crossings among all possible projections of G on the $\mathbb{R}^2$ plane. The investigation on the crossing number of graphs is a classical and however very difficult problem. The structure of Cartesian products of graphs makes Cartesian products of special graphs one of few graph classes for which the exact values of crossing numbers were obtained. (For a definition of Cartesian product, see [1].) Let C_n be the cycle on n vertices and P_m be the path on $m+1$ vertices. There are known exact values of crossing numbers for Cartesian products of paths with all graphs of order at most five as well as for Cartesian products of cycles and all graphs of order at most four. In addition, for some graphs G on five vertices the crossing numbers of $G \times C_n$ are known.

A drawing with the minimum number of crossings (an optimal drawing) must be a *good* drawing; that is, each two edges have at most one point in common, which is either a common end-vertex or a crossing. Let D be a good drawing of the graph G. We denote the number of crossings in D by $cr_D(G)$. Let G_i and G_j be edge–disjoint subgraphs of G. We denote by $cr_D(G_i, G_j)$ the number of crossings between edges of G_i and edges of G_j, and by $cr_D(G_i)$ the number of crossings among edges of G_i in D.

In the paper [5], Patil and Krishnnamurthy established family of graphs for which power graphs have crossing number one. This is the only result concerning crossing numbers of power of some graphs. Let P_m^2 denote the power of the path P_m. We start to determine crossing numbers of a new infinite family of graphs, concretely for the Cartesian products $P_m^2 \times C_n$ where $m \geq 2$ and $n \geq 3$.

Received: 13.11.2008. In revised form: 09.03.2009. Accepted: 20.05.2009.
2000 *Mathematics Subject Classification*. 05C10, 05C38.
Key words and phrases. *Graph, power of graph, drawing, crossing number, cycle, Cartesian product.*
[1]The research was supported by the Slovak VEGA grant No. 1/0636/08.
[2]This work was supported by the Slovak Research and Development Agency under the contract No. APVV-0073-07.

For $m = 2$, the graph $P_2^2 \times C_n$ is isomorphic to the graph $C_3 \times C_n$. Since $cr(C_3 \times C_n) = n$ [7], we have $cr(P_2^2 \times C_n) = n$. As the graph $P_3^2 \times C_n$ contains the graph $C_4 \times C_n$ as a subgraph and $cr(C_4 \times C_n) = 2n$, $n \geq 4$ [1], for all integers n greater than 3 we know that $cr(P_3^2 \times C_n) \geq 2n$. The drawing of the graph $P_3^2 \times C_n$ with $2n$ crossings shows that, for $n \geq 4$, $cr(P_3^2 \times C_n) \leq 2n$. This, together with the result $cr(P_3^2 \times C_3) = 6$ [2], confirms that $cr(P_3^2 \times C_n) = 2n$ for all $n \geq 3$. For the graph $P_4^2 \times C_n$ one can find the drawing with $3n$ crossings, hence $cr(P_4^2 \times C_n) \leq 3n$. As the graph $P_4^2 \times C_n$ contains the graph $C_5 \times C_n$ as a subgraph and, for $n \geq 5$, $cr(C_5 \times C_n) = 3n$ [4], we have that the crossing number of the graph $P_4^2 \times C_n$ is $3n$ for all $n \geq 5$. The graph P_4^2 is the special graph on five vertices and it is proved in [3] that $cr(P_4^2 \times C_3) = 9$ and $cr(P_4^2 \times C_4) = 12$. This confirms that $cr(P_4^2 \times C_n) = 3n$ for all $n \geq 3$.

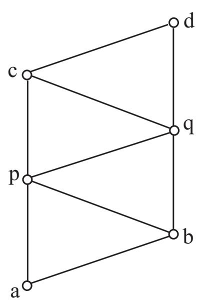
The extension of the drawing in Figure 1(b) shows that the crossing number of the Cartesian product $P_m^2 \times C_n$ is at most $n(m-2)$. The main result of the paper is that the crossing number of the graph $P_5^2 \times C_n$ is $4n$ for all $n \geq 3$.

2. The graph $P_5^2 \times C_n$

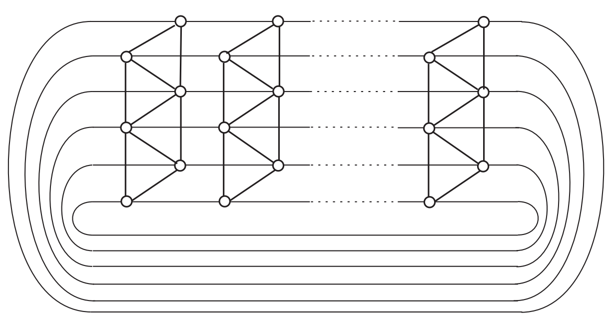
Let P_5 by the path of length five, that is P_5 has six vertices. Figure 1(a) shows the power graph P_5^2. For the simpler labelling let, in this paper, H denote the graph P_5^2.

We assume $n \geq 3$ and find it convenient to consider the graph $P_5^2 \times C_n$ in the following way: it has $6n$ vertices and edges that are the edges in the n copies H^i, $i = 1, 2, \ldots, n$, and in the six cycles of length n. For $i = 1, 2, \ldots, n$, let a_i and d_i be the vertices of H^i of degree two, b_i and c_i the vertices of degree three, and p_i and q_i the vertices of degree four as shown in Figure 1(a). Thus, for $x \in \{a, b, c, d, p, q\}$, the n–cycle C_n^x is induced by the vertices $x_1, x_2, \ldots, x_n$. Let T^a (T^d) be the subgraph of the graph $P_5^2 \times C_n$ consisting of the cycle C_n^a (C_n^d) together with the vertices of C_n^b and C_n^p (C_n^c and C_n^q) and of the edges joining C_n^a (C_n^d) with C_n^b and C_n^p (C_n^c and C_n^q). Denote by I^{xy}, $x, y \in \{b, c, p, q\}$, the subgraph of $P_5^2 \times C_n$ consisting of the vertices in $V(C_n^x) \cup V(C_n^y)$ and of the edges $\{x_i, y_i\}$ for all $i = 1, 2, \ldots, n$. It is not difficult to see that

$$P_5^2 \times C_n = T^a \cup C_n^b \cup C_n^p \cup I^{pb} \cup I^{pc} \cup I^{pq} \cup I^{qb} \cup I^{qc} \cup C_n^c \cup C_n^q \cup T^d.$$



(a)



(b)

Fig. 1. The graph P_5^2, and the Cartesian product $P_5^2 \times C_n$.

The main result of the paper is the next Theorem 2.1.

Theorem 2.1. $cr(P_5^2 \times C_n) = 4n$ *for* $n \geq 3$.

Proof. In Figure 1(b) there is the drawing of the graph $P_5^2 \times C_n$ with $4n$ crossings and therefore $cr(P_5^2 \times C_n) \leq 4n$. As for $n \geq 6$ the graph $P_5^2 \times C_n$ contains the subgraph $C_6 \times C_n$ and it was proved in [6] that $cr(C_6 \times C_n) = 4n$, the proof is done for all $n \geq 6$. The cases $n = 3, 4$ and 5 we prove as separate Theorems in the rest of the paper. □

Let us consider the graph $K_{1,1,2}$ in which r and s be the vertices of degree three and u and v be the vertices of degree two. The graph $P_5^2 \times C_n$ contains several subgraphs isomorphic to the graph $K_{1,1,2} \times C_n$. Special subgraphs of the graph $K_{1,1,2} \times C_n$ we will denote in the similar way as in the graph $P_5^2 \times C_n$.

Lemma 2.1. *Let D be a good drawing of the graph $K_{1,1,2} \times C_3$ in which every of the subgraphs I^{ru}, I^{rv}, I^{su}, I^{sv}, and I^{rs} has at most two crossings on its edges. Let for each pair $x, y \in \{r, s, u, v\}$ the 3-cycles C_3^x and C_3^y do not cross each other and let $cr_D(C_3^r \cup I^{rs} \cup C_3^s) = 1$. Then $cr_D(C_3^u \cup I^{ru} \cup I^{su}, C_3^v \cup I^{rv} \cup I^{sv}) \neq 0$.*

Proof. By hypothesis, the only crossing in the subgraph $C_3^r \cup I^{rs} \cup C_3^s$ can appear between two edges of I^{rs} or between an edge of I^{rs} and an edge of C_3^r or C_3^s. Suppose first that two edges, say $\{r_i, s_i\}$ and $\{r_j, s_j\}$, of I^{rs} cross each other. In this case, in D, the vertex-disjoint cycles $r_i u_i s_i r_i$ and $r_j v_j s_j r_j$ cross each other at least two times and the vertex-disjoint cycles $r_i v_i s_i r_i$ and $r_j u_j s_j r_j$ cross each other at least two times. As at most two crossings can appear on the edges $\{r_i, s_i\}$ and $\{r_j, s_j\}$, the edges of $I^{ru} \cup I^{su}$ cross the edges of $I^{rv} \cup I^{sv}$, and therefore $cr_D(C_3^u \cup I^{ru} \cup I^{su}, C_3^v \cup I^{rv} \cup I^{sv}) \neq 0$.

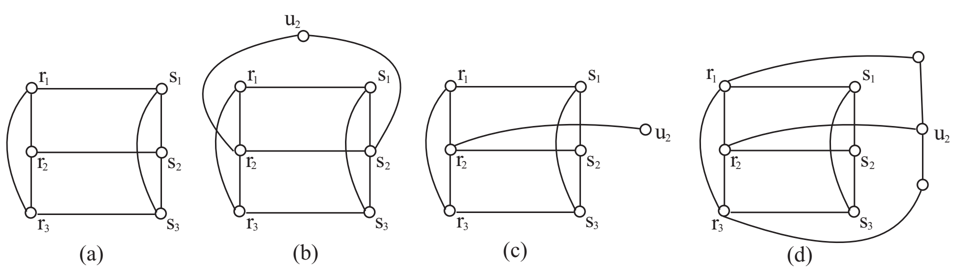


Fig. 2. The possible subdrawings of $C_3^r \cup I^{rs} \cup C_3^s$ and some edges of $C_3^u \cup I^{ru} \cup I^{su}$.

Consider now that, without loss of generality, an edge of I^{rs} crosses an edge of the 3-cycle C_3^s. The subdrawing D' induced from D by the subgraph $C_3^r \cup I^{rs} \cup C_3^s$ in unique up to the isomorphism, see Figure 2(a). The cycles C_3^u and C_3^v do not cross the edges of $C_3^r \cup I^{rs} \cup C_3^s$ and at most one of the subgraphs I^{ru}, I^{su}, I^{rv}, and I^{sv} can cross the edges of I^{rs}. Let, without loss of generality, $cr_D(I^{ru} \cup I^{su}, I^{rs}) = 0$. Since $cr_D(I^{ru}, C_3^r \cup C_3^s) \leq 2$ and $cr_D(I^{su}, C_3^r \cup C_3^s) \leq 2$, the cycle C_3^u can only lie in D in the unbounded region in the view of the subdrawing D'. The edge $\{u_2, s_2\}$ does not cross $C_3^r \cup I^{rs} \cup C_3^s$ and the edge $\{u_2, r_2\}$ can cross the cycle C_3^r once or it crosses the cycle C_3^s two times. In the first case we have the subdrawing of D shown in Figure 2(b). It is easily seen that, in D, both paths $r_1 v_1 v_2 r_2$ and $s_1 v_1 v_2 s_2$ cross the cycle $r_2 u_2 s_2 r_2$ and therefore at least one of them crosses the path $r_2u_2s_2$. In the second case, the edge $\{u_2, r_2\}$ crosses the cycle C_3^s two times as shown in Figure 2(c). Since the edges of I^{ru} are crossed two times and C_3^u does not cross the edges of $C_3^r \cup I^{rs} \cup C_3^s$, Figure 2(d) shows that the vertices s_1 and s_3 are separated in D by the edges of $C_3^r \cup I^{ru} \cup C_3^u$. The cycle C_3^v does not cross the edges of $C_3^r \cup I^{rs} \cup C_3^s \cup I^{ru} \cup C_3^u$ and therefore, in D, the cycle C_3^v is placed in one region of the subdrawing shown in Figure 2(d). It is easily seen that if, in D, the path $s_1v_1v_2s_2$ does not contain more than two crossings on the edges $\{s_1, v_1\}$ and $\{s_2, v_2\}$, then $cr_D(C_3^u \cup I^{ru} \cup I^{su}, C_3^v \cup I^{rv} \cup I^{sv}) \neq 0$. This completes the proof. □

Theorem 2.2. $cr(P_5^2 \times C_3) = 12$.

Proof. In Figure 1(b) it is possible to see that $cr(P_5^2 \times C_3) \leq 12$. To prove the reverse inequality assume that there is a drawing of the graph $P_5^2 \times C_3$ with fewer than 12 crossings and let D be such a drawing. The drawing D has the following properties:

Property 1. *The subgraph $C_3^p \cup I^{pq} \cup I^{pb}$ $(C_3^q \cup I^{pq} \cup I^{qc})$ has at most four crossings on its edges.*

Otherwise removing the edges of $C_3^p \cup I^{pq} \cup I^{pb}$ ($C_3^q \cup I^{pq} \cup I^{qc}$) from D results in the drawing of the subgraph homeomorphic to $K_{1,1,2}^s \times C_3$ with fewer than seven crossings, where $K_{1,1,2}^s$ is obtained from $K_{1,1,2}$ by an elementary subdivision of an edge of the 4-cycle. This is in contradiction with $cr(K_{1,1,2}^s \times C_3) = 7$, see [3].

Property 2. *The subgraph $C_3^b \cup I^{pb} \cup C_3^p \cup I^{pq}$ $(C_3^c \cup I^{qc} \cup C_3^q \cup I^{pq})$ has at most five crossings on its edges.*

Otherwise removing the edges of $C_3^b \cup I^{pb} \cup C_3^p \cup I^{pq}$ ($C_3^c \cup I^{qc} \cup C_3^q \cup I^{pq}$) results in the drawing of the graph homeomorphic to $K_{1,1,2} \times C_3$ with fewer than six crossings. This contradicts the fact that $cr(K_{1,1,2} \times C_3) = 6$, see [2].

Property 3. *The subgraph $C_3^b \cup I^{pb}$ $(C_3^c \cup I^{qc})$ has at most two crossings on its edges.*

Otherwise one can obtain the drawing of the subdivision of the graph $P_4^2 \times C_3$ with fewer than nine crossings, a contradiction with $cr(P_4^2 \times C_3) = 9$, see [3].

Property 4. *The subgraph T^a (T^d) has at most two crossings on its edges.*

Otherwise by deleting the edges of T^a (T^d) we have the drawing of the graph $P_4^2 \times C_3$ with fewer than nine crossings again.

Property 5. *In D there are at most two crossings on the edges of I^{pc} (I^{qb}).*

Otherwise deleting the edges of I^{pc} (I^{qb}) results in the drawing of the union of two graphs $C_3 \times C_3$ and $K_{1,1,2} \times C_3$ with one common C_3^p-cycle (C_3^q-cycle) with fewer than nine crossings. This contradicts the fact that such union of graphs has at least $3 + 6$ crossings, because none of crossings on the common 3-cycle appears in both graphs.

Property 6. *In D there are at most five crossings on the edges of $I^{pb} \cup I^{pq} \cup I^{qc}$.*

Otherwise D contains the subdrawing of $C_6 \times C_3$ with fewer than six crossings. This is in contradiction with $cr(C_6 \times C_3) = 6$, see [6].

Property 7. $cr_D(C_3^p \cup I^{pq} \cup C_3^q) \neq 0$, *and* $cr_D(C_3^p \cup I^{pb} \cup C_3^b) \neq 0$, *and* $cr_D(C_3^q \cup I^{qc} \cup C_3^c) \neq 0$.

If $cr_D(C_3^p \cup I^{pq} \cup C_3^q) = 0$, then none of crossings on the edges $C_3^p \cup I^{pq} \cup C_3^q$ appears in both subgraphs $T^a \cup C_3^b \cup I^{pb} \cup I^{qb} \cup C_3^p \cup I^{pq} \cup C_3^q$ and $T^d \cup C_3^c \cup I^{qc} \cup I^{pc} \cup C_3^p \cup I^{pq} \cup C_3^q$. As both these subgraphs are isomorphic with the graph $K_{1,1,2} \times C_3$ with crossing number six, D has more than eleven crossings on its edges. The similar contradiction can by obtained for $cr_D(C_3^p \cup I^{pb} \cup C_3^b) = 0$ ($cr_D(C_3^q \cup I^{qc} \cup C_3^c) = 0$) using subgraphs isomorphic with $P_4^2 \times C_3$ and $C_3 \times C_3$ which crossing numbers are nine and three, respectively.

In the next we show that $cr_D(C_3^p, C_3^q) = cr_D(C_3^p, C_3^b) = cr_D(C_3^p, C_3^c) = cr_D(C_3^b, C_3^q) = cr_D(C_3^c, C_3^q) = 0$. Assume that $cr_D(C_3^p, C_3^q) \neq 0$. In any good drawing, if two vertex-disjoint cycles cross each other, then at least one of them separates two vertices of the other. If C_3^p separates two vertices of C_3^q, say q_i and q_j, then C_3^p is crossed in D two times by C_3^q and by all paths $q_ib_ib_jq_j$, $q_ic_ic_jq_j$, and $q_id_id_jq_j$. This contradicts Property 1. The same contradiction is obtain when C_3^q separates two vertices of C_3^p and hence $cr_D(C_3^p, C_3^q) = 0$. Consider now that $cr_D(C_3^p, C_3^b) \neq 0$. The cycle C_3^b does not separate two vertices of C_3^p, otherwise it is crossed two times by C_3^p and by the path $p_ia_ia_jp_j$. This contradicts Property 3. If C_3^p separates two vertices b_i and b_j of C_3^b, then the cycle C_3^q is placed in D in the region with two vertices of C_3^b on its boundary and therefore one edge of I^{qb} crosses C_3^p. As C_3^p is also crossed by the path $b_ia_ia_jb_j$ and two times by C_3^b, Property 1 and the condition $cr_D(C_3^p, C_3^q) = 0$ imply that $cr_D(C_3^p \cup I^{pq} \cup C_3^q) = 0$ which contradicts Property 7. Thus, $cr_D(C_3^p, C_3^b) = 0$. The same arguments give $cr_D(C_3^q, C_3^c) = 0$, $cr_D(C_3^q, C_3^b) = 0$, and $cr_D(C_3^p, C_3^c) = 0$. This, together with Property 7, implies that there is at least one crossing on the edges of I^{pb} in the subgraph $C_3^b \cup I^{pb} \cup C_3^p$, at least one crossing on the edges of I^{pq} in the subgraph $C_3^p \cup I^{pq} \cup C_3^q$, and at least one crossing on the edges of I^{qc} in the subgraph $C_3^q \cup I^{qc} \cup C_3^c$.

Assume now that in the drawing D there are more than two crossings on the edges of I^{pq}. Then, by Property 6, $cr_D(C_3^p \cup I^{pb} \cup C_3^b) = 1$ and $cr_D(C_3^q \cup I^{qc} \cup C_3^c) = 1$. As $cr(C_3 \times C_3) = 3$, the subgraph T^a has its edges crossed at least two times. Thus, removing the edges of T^a and I^{pq} from D results in the drawing of the graph $K_{1,1,2}^s \times C_3$ with fewer than seven crossings, a contradiction. So, I^{pq} has at most two crossings on its edges and every of the subgraphs $T^a \cup C_3^b \cup I^{pb} \cup I^{qb} \cup C_3^p \cup I^{pq} \cup C_3^q$, $T^d \cup C_3^c \cup I^{qc} \cup I^{pc} \cup C_3^p \cup I^{pq} \cup C_3^q$, and $C_3^p \cup I^{pq} \cup I^{pb} \cup I^{pc} \cup C_3^q \cup I^{qc} \cup I^{qb} \cup C_3^b \cup C_3^c$ is in compliance with the assumptions of Lemma 2.1 that every of the subgraphs I^{xy}, $x, y \in \{a, b, c, d, p, q\}$, has at most two crossings. It implies from Properties 6 and 7 that in at least one of the subgraphs $C_3^p \cup I^{pq} \cup C_3^q$, $C_3^p \cup I^{pb} \cup C_3^b$, and $C_3^q \cup I^{qc} \cup C_3^c$ exactly one crossing appears among its edges. If $cr_D(C_3^p \cup I^{pq} \cup C_3^q) = 1$, then only one common crossing appears in both subgraphs $T^a \cup C_3^b \cup I^{pb} \cup I^{qb} \cup C_3^p \cup I^{pq} \cup C_3^q$ and $T^d \cup C_3^c \cup I^{qc} \cup I^{pc} \cup C_3^p \cup I^{pq} \cup C_3^q$. As $cr(K_{1,1,2} \times C_3) = 6$ and, by Lemma 2.1, $cr_D(C_3^b \cup I^{pb} \cup I^{qb}, C_3^c \cup I^{qc} \cup I^{pc}) \neq 0$, every of these subgraphs has its edges crossed at least seven times. Hence, in D there are at least $7 + 7 - 2 = 12$ crossings, a contradiction. The similar arguments in the case when $cr_D(C_3^p \cup I^{pb} \cup C_3^b) = 1$ or $cr_D(C_3^q \cup I^{qc} \cup C_3^c) = 1$ together with the facts that $cr(C_3 \times C_3) = 3$ and $cr(P_4^2 \times C_3) = 9$ gives the same contradiction. This completes the proof. □

Note that for $n \geq 4$ there is no good drawing of the subgraph $C_n^p \cup I^{pq} \cup C_n^q$ ($C_n^p \cup I^{pb} \cup C_n^b$, $C_n^q \cup I^{qc} \cup C_n^c$) with one crossing. In fact, if any two edges of the graph $C_n^p \cup I^{pq} \cup C_n^q$ not incident with the same vertex cross each other, then one can find two vertex-disjoint cycles in such a way that every of these cycles contains exactly one of the considered edges. As two vertex-disjoint cycles cannot cross each other exactly once, in the drawing there is one additional crossing.

Theorem 2.3. $cr(P_5^2 \times C_4) = 16.$

Proof. It is easy to see in Figure 1(b) that $cr(P_5^2 \times C_4) \leq 16$. To prove the reverse inequality assume that there is a drawing of the graph $P_5^2 \times C_4$ with fewer than 16 crossings and let D be such a drawing. As $cr(K_{1,1,2}^s \times C_4) = 12$ and $cr(C_6 \times C_4) = 12$, see [1] and [3], in a similar way as in the proof of Theorem 2.2 it is easily seen that D has the following properties:

Property 1. *The subgraph* $C_4^p \cup I^{pq} \cup I^{pb}$ $(C_4^q \cup I^{pq} \cup I^{qc})$ *has at most three crossings on its edges.*

Property 2. *The subgraph* $C_4^b \cup I^{pb} \cup I^{pq}$ $(C_4^c \cup I^{qc} \cup I^{pq})$ *has at most three crossings on its edges.*

Property 3. *The subgraph* $I^{pb} \cup I^{pq} \cup I^{qc}$ *has at most three crossings on its edges.*

Using the same arguments as in the proof of Theorem 2.2, one can prove the next fact:

Property 4. $cr_D(C_4^x, C_4^y) = 0$ for all $x, y \in \{b, c, p, q\}$.

If $cr_D(C_4^p \cup I^{pq} \cup C_4^q) = 0$, none of crossing among edges of the subgraph $T^a \cup C_4^b \cup I^{pb} \cup I^{qb} \cup C_4^p \cup I^{pq} \cup C_4^q$ is a crossing in the subgraph $T^d \cup C_4^c \cup I^{qc} \cup I^{pc} \cup C_4^p \cup I^{pq} \cup C_4^q$ and vice versa. In this case, as $cr(K_{1,1,2} \times C_4) = 8$, D has at least $8+8 = 16$ crossings, a contradiction. Using $cr(C_3 \times C_4) + cr(P_4^2 \times C_4) = 4+12$, the same contradiction is obtained if $cr_D(C_4^p \cup I^{pb} \cup C_4^b) = 0$ or $cr_D(C_4^q \cup I^{qc} \cup C_4^c) = 0$. This proves the following:

Property 5. $cr_D(C_4^p \cup I^{pq} \cup C_4^q) \geq 2$, $cr_D(C_4^p \cup I^{pb} \cup C_4^b) \geq 2$ *and* $cr_D(C_4^q \cup I^{qc} \cup C_4^c) \geq 2$.

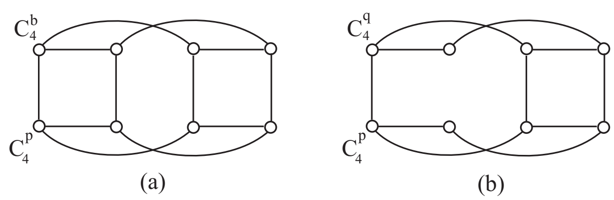


Fig. 3. The possible subdrawings of $C_4^p \cup I^{pb} \cup C_4^b$ and $C_4^p \cup I^{pq} \cup C_4^q$.

Consider now that in the subdrawing of $C_4^p \cup I^{pb} \cup C_4^b$ there is no crossing on the edges of I^{pb}. As $cr_D(C_4^p \cup I^{pb} \cup C_4^b) \geq 2$ and $cr_D(C_4^p, C_4^b) = 0$, every of the cycles C_4^p and C_4^b has an internal crossing and the unique subdrawing of $C_4^p \cup I^{pb} \cup C_4^b$ is shown in Figure 3(a). In this case $cr_D(C_4^q, I^{pb}) = 0$, otherwise both $C_4^p \cup I^{pq} \cup I^{pb}$ and $C_4^b \cup I^{pb} \cup I^{pq}$ have three crossings in the subdrawing of $C_4^p \cup I^{pq} \cup I^{pb} \cup C_4^b \cup I^{qb} \cup C_4^q$ and another crossings with the edges of T^a. This contradicts Property 1 and Property 2. Hence, the cycle C_4^q does not cross in D the edges of $C_4^p \cup I^{pb} \cup C_4^b$ and in Figure 3(a) it is easy to verify that in D the edges of I^{pq} cross at least two times the edges of $C_4^p \cup I^{pb} \cup C_4^b$. Now a path joining a vertex of C_4^p with a vertex of C_4^b containing a vertex of C_4^a crosses the edges of $C_4^p \cup I^{pb} \cup C_4^b$ and we have contradiction with Properties 1 and 2 again. Thus, we conclude that in the subdrawing of $C_4^p \cup I^{pb} \cup C_4^b$ an edge of I^{pb} is crossed as well as in the subdrawing of $C_4^q \cup I^{qc} \cup C_4^c$ an edge of I^{qc} is crossed. Property 3 implies that in D there is at most one crossing on the edges of I^{pq}.

In the drawing D there are at least nine crossings among the edges of the subgraph $T^a \cup C_4^b \cup I^{pb} \cup I^{qb} \cup C_4^p \cup I^{pq} \cup C_4^q$ as well as among the edges of the subgraph $T^d \cup C_4^c \cup I^{qc} \cup I^{pc} \cup C_4^p \cup I^{pq} \cup C_4^q$, because both I^{pb} and I^{qc} are crossed and $cr(C_4 \times C_4) = 8$. This follows that in the subdrawing of the graph $C_4^p \cup I^{pq} \cup C_4^q$ there are at least three crossings among its edges. Otherwise at most two crossings of $C_4^p \cup I^{pq} \cup C_4^q$ are counted in both subgraphs $T^a \cup C_4^b \cup I^{pb} \cup I^{qb} \cup C_4^p \cup I^{pq} \cup C_4^q$ and $T^d \cup C_4^c \cup I^{qc} \cup I^{pc} \cup C_4^p \cup I^{pq} \cup C_4^q$ and D has at least $9+9-2=16$ crossings, a contradiction. Hence, as $cr_D(C_4^p, C_4^q) = 0$ and on the edges of I^{pq} there is only one crossing, the edges of $C_4^p \cup I^{pq} \cup C_4^q$ cross each other in such a way that the cycle C_4^p has an internal crossing and the cycle C_4^q has an internal crossing. The deleting of a crossed edge of I^{pq} gives the unique subdrawing shown in Figure 3(b). Properties 1 and 2 allow only two another crossings on the edges of the subdrawing in Figure 3(b) and it is easy to see that it is impossible to place the rest of the edges of the graph to obtain our considered drawing D. This completes the proof of Theorem 2.3. □

Theorem 2.4. $cr(P_5^2 \times C_5) = 20$.

Proof. Figure 1(b) shows that $cr(P_5^2 \times C_5) \leq 20$. To prove the reverse inequality assume that there is a drawing of the graph $P_5^2 \times C_5$ with fewer than 20 crossings and let D be such a drawing. As $cr(C_5 \times C_5) = 15$, every subgraph $C_5^b \cup I^{pb} \cup I^{pq} \cup I^{qc}$, $I^{pb} \cup C_5^p \cup I^{pq} \cup I^{qc}$, $I^{pb} \cup I^{pq} \cup C_5^q \cup I^{qc}$ and $I^{pb} \cup I^{pq} \cup I^{qc} \cup C_5^c$ has in D at most four crossings on its edges. Moreover, as $cr(C_6 \times C_5) = 18$, on the edges of $I^{pb} \cup I^{pq} \cup I^{qc}$ there is at most one crossing. Using these restrictions one can show that $cr_D(C_5^p, C_5^q) = 0$ and that if two cycles C_5^x and C_5^y, $x \in \{p,q\}, y \in \{b,c\}$ cross each other, then the cycle C_5^x does not have an internal crossing.

We know that if the edges of the subgraph $C_5^p \cup I^{pq} \cup C_5^q$ ($C_5^p \cup I^{pb} \cup C_5^b$, $C_5^q \cup I^{qc} \cup C_5^c$) cross each other in D, then they cross each other at least two times. Using the similar arguments as in the proof of Theorem 2.4, one can show that the condition $cr_D(C_5^p \cup I^{pq} \cup C_5^q) = 0$ ($cr_D(C_5^p \cup I^{pb} \cup C_5^b) = 0$, $cr_D(C_5^q \cup I^{qc} \cup C_5^c) = 0$) contradicts the assumption of the drawing D. Hence, $cr_D(C_5^p \cup I^{pq} \cup C_5^q) \geq 2$, $cr_D(C_5^p \cup I^{pb} \cup C_5^b) \geq 2$ and $cr_D(C_5^q \cup I^{qc} \cup C_5^c) \geq 2$.

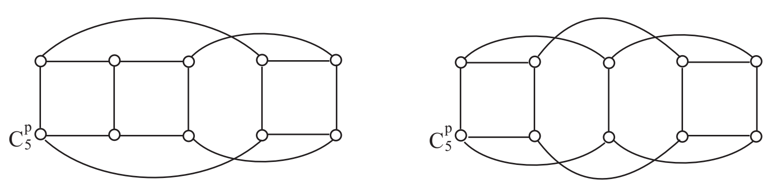


Fig. 4. The possible subdrawings of $C_5^p \cup I^{pq} \cup C_5^q$.

Consider first that in D there is no crossing on the edges of I^{pq}. In this case every of C_5^p and C_5^q has exactly one internal crossing or every of C_5^p and C_5^q has exactly two internal crossings. The only possible subdrawings of $C_5^p \cup I^{pq} \cup C_5^q$ induced from D are shown in Figure 4. Since both cycles C_5^p and C_5^q have internal crossings, it is clear that $cr_D(C_5^p, C_5^b) = cr_D(C_5^p, C_5^c) = cr_D(C_5^q, C_5^b) =$

$cr_D(C_5^q, C_5^c) = 0$. In Figure 4 it is easily seen that if C_5^c is placed in one of the regions of the subdrawing of $C_5^p \cup I^{pq} \cup C_5^q$, then the edges of I^{qc} are crossed more than once, a contradiction.

If in D there is a crossing (exactly one) on the edges of I^{pq}, then there is no crossing on the edges of I^{pb} in the subdrawing of $C_5^p \cup I^{pb} \cup C_5^b$, and the similar consideration as in the previous case leads to the same contradiction with the assumption on the drawing D. This completes the proof. □

REFERENCES

[1] Beineke, L. W. and Ringeisen, R. D., *On the crossing numbers of products of cycles and graphs of order four*, J. Graph Theory **4** (1980), 145-155

[2] Klešč, M., *The crossing number of* $(K_4 - e) \times C_3$, Proceedings of the International Scientific Conference on Mathematics, Herľany 1999, 106-109, Univ. Technol. Košice, Košice, 2000

[3] Klešč, M., *Some crossing numbers of products of cycles* , Discuss. Math. Graph Theory **25** (2005), 197-210

[4] Klešč, M., Richter, R. B. and Stobert, I., *The crossing number of* $C_5 \times C_n$, J. Graph Theory **22** (1996), 239-243

[5] Patil, H. P. and Krishnnamurthy, D., *On power graphs with crossing number one*, Discuss. Math. **12** (1992), 27-37

[6] Richter, R. B. and Salazar, G., *The crossing number of* $C_6 \times C_n$, Australasian Journal of Combinatorics **23** (2001), 135-144

[7] Ringeisen, R. D. and Beineke, L. W., *The crossing number of* $C_3 \times Cn$, J. Combin. Theory **24** (B) (1978), 134-136

TECHNICAL UNIVERSITY OF KOŠICE
DEPARTMENT OF MATHEMATICS
FACULTY OF ELECTRICAL
ENGINEERING AND INFORMATICS
0420 KOŠICE, SLOVAK REPUBLIC
E-mail address: `Marian.Klesc@tuke.sk`
E-mail address: `Daniela.Kravecova@tuke.sk`