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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} anniversary

On some existence and uniqueness theorems for Fredholm and Volterra equations with modified argument

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ABSTRACT. In this paper some existence and uniqueness theorems for a Fredholm, respectively, a Volterra integral equation, are given by using the contraction mapping principle and the generalized contraction principle, respectively. These integral equations arise in several concrete applications such as theory of optimal control, economics and etc.

1. INTRODUCTION

In the paper [7], the authors study a Volterra type integral equation of the form

$$x(t) = g(t, x(t)) + f\left(t, \int_0^t K(t, s, x(s))ds, x(\alpha(t))\right) \tag{1.1}$$

using Darbo's fixed point theorem and several others concepts, like the Kuratowski measure of noncompactness. Their main result may be stated as follows:

Theorem 1.1. *Assume that the following conditions are satisfied:*

(H1) $g : [0, a] \times \mathbb{R} \to \mathbb{R}$ *and* $f : [0, a] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous and there exist nonnegative constants* μ, γ, λ *such that*

$$|g(t, 0)| \leq \mu,$$
$$|f(t, 0, x(\alpha(t)))| \leq \gamma + \lambda \cdot |x(t)|$$

for $t \in [0, a]$.

(H2) there exist the continuous functions $a_1, a_2, a_3 : [0, a] \to [0, a]$ *such that*

$$|g(t, x_1) - g(t, x_2)| \leq a_1(t) |x_1 - x_2|,$$
$$|f(t, y_1, x) - f(t, y_2, x)| \leq a_2(t) |y_1 - y_2|,$$
$$|f(t, y, x_1) - f(t, y, x_2)| \leq a_3(t) |x_1 - x_2|,$$

for all $x_i, y_i \in \mathbb{R}$ *,*$i = 1, 2$ *and* $t \in [0, a]$ *and let* $k = \max\limits_j \{|a_j(t)| : t \in [0, a]\}$ *for* $j = 1, 2, 3$.

(H3) $K(t, s, x) : [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies a sublinear condition, that is there exist the constant* α *and* β *such that*

$$|K(t, s, x)| \leq \alpha + \beta |x|,$$

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for all $t, s \in [0, a]$ *and* $x \in \mathbb{R}$

(H4)

$$k \leq \frac{1-\lambda}{2(1+a\beta)}$$

Then equation (1.1) has at least one solution in the Banach algebra $C[0, a]$.

Starting from this theorem, in the present paper, we obtain a similar result for a simpler Fredholm integral equation of the form:

$$x(t) = f(t, x(t)) + \int_0^a K(t, s, x(s))ds, \ t \in [0, a] \tag{1.2}$$

using the mapping contraction principle. For other similar existence and uniqueness results for Fredholm integral equations based on the technique of Picard operators or weakly Picard operators, see for example, the recent papers [4], [5] and references therein.

2. An existence and uniqueness theorem for a Fredholm integral equation

In this section we recall some basic results which we will need in the following section.

Let (X, d) be a metric space, where X be a nonempty set, $T : X \to X$ an operator and let us denote $F_T := \{x \in X / Tx = x\}$, the fixed point set of T.

Definition 2.1. An operator $T : X \to X$ is called a *Picard operator* if then exists $x^* \in X$ such that

(i) $F_T = \{x^*\}$;

(ii) the sequence $\{T^n(x_0)\}_{n \in N}$ converges to x^*, for all $x_0 \in X$.

Definition 2.2. A mapping $T : X \to X$ is said to be:

(i) *Lipschitzian* if there exist $L > 0$ such that $d(Tx, Ty) \leq L \cdot d(x, y)$, for all $x, y \in X$;

(ii) *contraction* if it is Lipschitzian with $L < 1$.

The next theorem is the main tool used in this paper, see for example [2].

Theorem 2.2. (Contraction mapping principle) *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a given a-contraction, that is an operator satisfying*

$$d(Tx, Ty) \leq ad(x, y),$$

for any $x, y \in X$ *with* $a \in [0, 1)$ *fixed.*

Then

(i) T has a unique fixed point x^*, *that is,* $F_T = x^*$;

(ii) the Picard iteration associated to T, i.e., the sequence $\{x_n\}_{n \geq 0}$, *defined by*

$$x_n = T(x_{n-1}) = T^n(x_0), \ n = 1, 2, \ldots$$

converge to x^*, *for any initial guess* $x_0 \in X$;

(iii) *The priori and a posteriori error estimates*

$$d(x_n, x^*) \leq \frac{a^n}{1-a} \cdot d(x_0, x_1),\ n = 0, 1, 2, ...$$

$$d(x_n, x^*) \leq \frac{a}{1-a} \cdot d(x_{n-1}, x_n),\ n = 0, 1, 2, ...$$

hold.

(iv) *The rate of convergence is given by*

$$d(x_n, x^*) \leq a \cdot d(x_{n-1}, x^*) \leq a^n \cdot d(x_0, x^*),\ n = 1, 2, ...$$

Now, we shall study the integral equation (1.2) and we shall establish a result concerning the existence and uniqueness of solutions of this equation in the set $C[0,a]$.

Assume that the following assumptions are satisfied:

(i) $f : [0,a] \times \mathbb{R} \to \mathbb{R},\ K : [0,a] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions;

(ii) there exist the continuous functions $a_1, a_2 : [0,a] \to \mathbb{R}_+$ such that:

$$|f(t,x_1) - f(t,x_2)| \leq\ a_1(t)\,|x_1 - x_2|\,,$$

$$|K(t,s,x_1) - K(t,s,x_2)| \leq\ a_2(t)\,|x_1 - x_2|\,;$$

(iii) there exist the real numbers k_1, k_2 such that $a_1(t) \leq k_1, a_2(t) \leq k_2$, for $t \in [0,a]$;

(iv) $k_1 + a \cdot k_2 < 1$.

Theorem 2.3. *Under the assumptions (i)-(iv) above the equation (1.2) has a unique solution in $C[0,a]$ and the iterative approximations sequence associated to the Fredholm operator that is,*

$$x_{n+1}(t) = f(t, x_n(t)) + \int_0^a K(t,s,x_n(s))ds,\ n \geq 0$$

converge to $x^*, \forall x_0 \in C[0,a]$, and we have the estimate

$$\|x_n - x^*\| \leq \frac{(k_1 + a \cdot k_2)^n}{1 - k_1 - a \cdot k_2} \cdot \|x_1 - x_0\|\,, n \geq 1.$$

Proof. The proof of this result uses Theorem 2.2 as the main tool. We define the operator H on the space $C[0,a]$, in the following way: $H : C[0,a] \to C[0,a]$,

$$(Hx)(t) = f(t,x(t)) + \int_0^a K(t,s,x(s))ds\ t \in [0,a].$$

Let us fix $x, y \in C[0,a]$, then using our assumptions for $t \in [0,a]$, we get:

$$|(Hx)(t) - (Hy)(t)| = \left| f(t,x(t)) + \int_0^a K(t,s,x(s))ds - f(t,y(t)) - \int_0^a K(t,s,y(s))ds \right|$$

$$\leq |f(t,x(t)) - f(t,y(t))| + \int_0^a |K(t,s,x(s)) - K(t,s,y(s))|\,ds$$

$$\leq a_1(t)\cdot|x(t)-y(t)|+a_2(t)\cdot\int_0^a|x(s)-y(s)|\,ds < k_1\cdot|x(t)-y(t)|+k_2\cdot\int_0^a|x(s)-y(s)|\,ds$$

Applying the norm in the previous inequality, we obtain

$$\|Hx-Hy\| < (k_1+a\cdot k_2)\,\|x-y\|\,.$$

In view of the assumptions above we have:

$$\|Hx-Hy\| < L\cdot\|x-y\|\,,\ L=k_1+a\cdot k_2<1,$$

so the operator H is a contraction and the conclusion follows by Theorem 2.2. □

Theorem 2.4. *If the hypothesis (ii) in Theorem 2.3 is replaced by*

(ii‘) there exist the continuous functions $a_1,a_2:[0,a]\to\mathbb{R}_+$ *such that*

$$|f(t,x_1)-f(t,x_2)|\leq\ a_1(t)\,|x_1-x_2|$$

$$|K(t,s,x_1)-K(t,s,x_2)|\leq\ a_2(s)\,|x_1-x_2|\,,$$

for any $t,s\in[0,a]$
and (iii) is replaced by

(iii‘) there exists a real number k, such that

$$a_1(t)+\int_0^a a_2(s)ds\leq k<1,$$

then the equation (1.2) has a unique solution in $C[0,a]$.

Proof. With H in the proof of theorem 2.3, we have for $x,y\in C[0,a]$:

$$|(Hx)(t)-(Hy)(t)|=\left|f(t,x(t))+\int_0^a K(t,s,x(s))ds-f(t,y(t))-\int_0^a K(t,s,y(s))ds\right|$$

$$\leq|f(t,x(t))-f(t,y(t))|+\int_0^a|K(t,s,x(s))-K(t,s,y(s))|\,ds$$

$$\leq a_1(t)\cdot|x(t)-y(t)|+\int_0^a a_2(s)\,|x(s))-y(s)|\,ds$$

Using the norm in the previous inequality and the assumption (iii‘), we get:

$$\|Hx-Hy\|\leq\left(a_1(t)+\int_0^a a_2(s)ds\right)\cdot\|x-y\|\leq k\cdot\|x-y\|$$

Then H is a contraction operator and now apply again Theorem 2.2. □

Remark 2.1. Similary to Theorem 2.3 the iterative approximations sequence, that is,

$$x_{n+1}(t) = f(t, x_n(t)) + \int_0^a K(t, s, x_n(s))ds, \ n \geq 0$$

converge to $x^*, \forall x_0 \in C[0, a]$, and we have the estimate

$$\|x_n - x^*\| \leq \frac{k^n}{1-k} \cdot \|x_1 - x_0\|, n \geq 1.$$

3. An existence and uniqueness theorem for a Volterra integral equation

Definition 3.1. A function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies:

(i) ϕ is monotone increasing $(t_1 \leq t_2 \Rightarrow \phi(t_1) \leq \phi(t_2))$;
(ii) $\phi^n(t) \to 0$ for any $t \geq 0$;

is called a *comparison function*.

Definition 3.2. A function ϕ which satisfies:

(i) ϕ is monotone increasing $(t_1 \leq t_2 \Rightarrow \phi(t_1) \leq \phi(t_2))$;
(ii) $\sum\limits_{i=1}^{\infty} \phi^k(t)$ is convergent for $t \geq 0$ is called *c - comparison function*.

Definition 3.3. Let (X, d) metric space. A application $T : X \to X$ is said to be a ϕ - *contraction* if there exist a comparison function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d(Tx, Ty) \leq \phi(d(x, y))$$

for any $x, y \in X$

The following theorems and corollaries were given in [2].

Theorem 3.1. *Let us consider (X, d) is a metric space and the map $T : X \to X$ is a ϕ - contraction. Then T is an Picard operator.*

Corollary 3.1. *Let us suppose (X, d) is a complete metric space and the map $T : X \to X$ for which $\exists k \in \mathbb{N}^*$ such that T^k is a ϕ - contraction. Then $F_T = \{x^*\}$.*

Theorem 3.2. *Let (X, d) be a complete metric space and $T : X \to X$ is a ϕ - contraction, with ϕ an c- comparison function. Then*

(1) $F_T = \{x^*\}$;
(2) the Picard iteration $x_n = \{T^n x_0\}_{n \in \mathbb{N}}$ converge to x^* as $n \to \infty$ for any $x_0 \in X$;
(3) $d(x_n, x^*) \leq s(d(x_n, x_{n+1}))$, $n = 0, 1, 2...$
where $s(t) = \sum\limits_{i=1}^{\infty} \phi^k(t)$ is the sum of the series of comparison.

Now, we shall study the integral equation with modified argument

$$x(t) = f(t) + \lambda \int_{-t}^{t} K(t, s, x(s), x(g(s)))ds \tag{3.1}$$

for $t \in [-T, T]$, $T > 0$, $\lambda \in \mathbb{R}_+$.

We shall establish one result concerning the existence and uniqueness of solution of this equation in $C[-T,T]$, using the ϕ- contraction principle. This equation has been studied in [4], [5] using the Chebisev norm. In this paper, our result shall be obtained by using a Bielecki type norm.

Assume that the following conditions are satisfied:

(i) $K \in C([-T,T] \times [-T,T] \times \mathbb{R}^2)$

(ii) $f \in C[-T,T]$

(iii) $K(x,s,\bullet,\bullet) : [-T,T] \times [-T,T] \to \mathbb{R}$ increasing for any $x,s \in [-T,T]$

(iv) there exists the comparison function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\phi(\alpha t) \leq \alpha\phi(t)$ for any $t \in [-T,T]$, $\alpha \geq 1$ such that

$$|K(x,s,u_1,v_1) - K(x,s,u_2,v_2)| \leq \phi(|u_1-u_2| + |v_1-v_2|)$$

for any $t,s \in [-T,T]$, $u_1,u_2,v_1,v_2 \in \mathbb{R}$, $u_i \leq v_i, i = 1,2$

Theorem 3.3. *Suppose* (i)-(iv) *are satisfied. Then the integral equation* (3.1) *has a unique solution x^* in $C[-T,T]$ and the iterative approximations sequences, defined by*

$$x_{n+1}(t) = f(t) + \lambda \int_{-t}^{t} K(t,s,x_n(s),x_n(g(s)))ds$$

converges to x^*, for each $x_0 \in C[-T,T]$,
and the following estimate

$$\|x_n - x^*\| \leq \frac{\left(\frac{2\cdot|\lambda|}{\tau}\right)^n}{1 - \frac{2\cdot|\lambda|}{\tau}} \cdot \|x_1 - x_0\|$$

holds.

Proof. We attach to the integral equation (3.1) the operator $A : C[-T,T] \to C[-T,T]$, defined by:

$$(Ax)(t) := f(t) + \lambda \int_{-t}^{t} K(t,s,x(s),x(g(s)))ds \ t \in C[-T,T].$$

We consider the Bielecki norm $\|x\|_B = \max_{t\in[-T,T]} \left|x(t)e^{-\tau(t-T)}\right|$, $\tau > 0$. The set of the solutions of the integral equation (3.1) coincides with the set of fixed points of the operator A.

By (iv) we have:

$$|(Ax_1)(t) - (Ax_2)(t)| \leq |\lambda| \int_{-t}^{t} |K(t,s,x_1(s),x_1(g(s))) - K(t,s,x_2(s),x_2(g(s)))|\, ds$$
$$= |\lambda| \int_{-t}^{t} |K(t,s,x_1(s),x_1(g(s))) - K(t,s,x_2(s),x_2(g(s)))| \cdot e^{-\tau(t-T)} \cdot e^{\tau(t-T)}ds.$$

Then

$$\begin{aligned} |Ax_1 - Ax_2| &\leq \tfrac{|\lambda|\cdot 2}{\tau} \cdot \phi(\|x_1 - x_2\|)(e^{\tau(t-T)} - e^{-\tau(t+T)}) \\ &\leq \tfrac{|\lambda|\cdot 2}{\tau} \cdot \phi(\|x_1 - x_2\|) \cdot e^{\tau(t-T)}. \end{aligned}$$

So,

$$|Ax_1 - Ax_2| \cdot e^{-\tau(t-T)} \leq \frac{|\lambda| \cdot 2}{\tau} \cdot \phi(\|x_1 - x_2\|).$$

Applying maximum in inequalities, we obtained:

$$\|Ax_1 - Ax_2\| \leq \frac{|\lambda| \cdot 2}{\tau} \cdot \phi(\|x_1 - x_2\|)$$

In this case A is an α- Lipschitzian operator with $\alpha = \frac{2\cdot|\lambda|}{\tau}$. If we take τ such that $\frac{2\cdot|\lambda|}{\tau} < 1 \Leftrightarrow |\lambda| < \frac{\tau}{2}$, then A is α- contraction and applying the Contraction Mapping Principle, equation (3.1) has a unique solution. □

4. Applications

In this section we present some examples of classical integral and functional equations considered in nonlinear analysis which are particular cases of (1.2).

Example 4.1. Let us take $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $K : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(t, x(t)) = \sin \frac{a}{a+t},$$
$$K(t, s, x(s)) = \frac{ats}{9} \cdot \sin x(s), a \in [0,1]$$

These functions are continuous and satisfy hypothesis (i)-(iv) with $a_1 = 0, a_2 = \dfrac{a}{9}$, $\max\{a_1 + a \cdot a_2\} = \frac{1}{9} \leq 1$. Applying the result obtained in Theorem 2.3 ,we deduce that the equation (1.2) has a unique solution in $C[0,1]$ which can be obtained by the sequence of successive approximation

$$x_{n+1}(t) = f(t, x_n(t)) + \int_0^a K(t, s, x_n(s))ds,\ n \geq 0$$

For $x_0 \equiv 0$, we get $K(t, s, x_0(s)) = 0$ and $x_1 = f(t, 0)$.

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