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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} anniversary

Fixed points for non-self nonlinear contractions and non-self Caristi type operators

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ABSTRACT. The purpose of this paper is to discuss some basic problems of the fixed point theory for non-self singlevalued operators on a set with two metrics. The results complement and extend some known results in the literature.

1. INTRODUCTION

Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$. Denote by $B_d(x_0; r) := \{x \in X | d(x_0, x) < r\}$ the open ball centered in x_0 with radius r and by $\widetilde{B}_d(x_0; r) := \{x \in X | d(x_0, x) \le r\}$ the closed ball centered in x_0 with radius r. If ρ is another metric on X, then we denote be $\bar{B}_d^{\rho}(x_0, r)$ the closure of the ball $B_d(x_0; r)$ with respect to ρ.

Let $f : X \to X$ be an operator. Then, $x^* \in X$ is called a fixed point for f if $x^* = f(x^*)$. Denote by $Fixf := \{x \in X | x = f(x)\}$ the fixed point set of f. Also, we denote by $I(f) := \{Y \subseteq X | f(Y) \subset Y\}$ the set of all invariant subsets for f and by $I_b(f) := \{Y \in I(f) | Y \text{ is bounded}\}$ the set of all bounded invariant subsets for f.

An operator $f : Y \subseteq X \to X$ is said to be an a-contraction if $a \in [0, 1[$ and $d(f(x), f(y)) \le ad(x, y)$, for all $x, y \in Y$.

The following result is an easy consequence of the Caccioppoli

Theorem 1.1. *(see Dugundji-Granas* [3]*, pp. 11) Let (X, d) a complete metric space, $x_0 \in X$ and $r > 0$. If $f : B(x_0; r) \to X$ is an a-contraction and $d(x_0, f(x_0)) < (1-a)r$, then f has a unique fixed point.*

Let us remark that if $f : \widetilde{B}(x_0; r) \to X$ is an a-contraction such that $d(x_0, f(x_0)) \le (1 - a)r$, then $\widetilde{B}(x_0; r) \in I(f)$ and again f has a unique fixed point in $\widetilde{B}(x_0; r)$.

Let E be a Banach space and $Y \subset E$. Given an operator $f : Y \to E$, the operator $g : Y \to E$ defined by $g(x) := x - f(x)$ is called the field associated with f. An operator $f : Y \to E$ is said to be open if for any open subset U of Y the set $f(U)$ is open in E too.

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As a consequence of the above result, one obtains the following domain invariance theorem for contraction type fields.

Theorem 1.2. *(see Dugundji-Granas* [3], *pp. 11) Let E be a Banach space and Y be an open subset of E. Consider $f : U \to E$ be an a-contraction. Let $g : U \to E$ $g(x) := x - f(x)$, the associated field. Then:*

(a) $g : U \to E$ is an open operator;

(b) $g : U \to g(U)$ is a homeomorphism. In particular, if $f : E \to E$, then the associated field g is a homeomorphism of E into itself.

The purpose of this paper is to discuss some basic problems of the fixed point theory for non-self singlevalued generalized contractions on a set with two metrics. The results complement and extend some known results in the literature, see [1], [2], [4], [7], [8]. [9].

2. Preliminaries

Throughout the paper, by $\mathbb{R}_+$ we denote the set of all real nonnegative numbers, while $\mathbb{N}$ is the set of all natural numbers. Also, $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

Recall that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function (see [7]) if it is increasing and $\varphi^k(t) \to 0$, as $k \to +\infty$. As a consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and φ is continuous in 0.

Recall also the notion of strict comparison function. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strict comparison function (see [7]) if it is strictly increasing and $t - \varphi(t) \to +\infty$ when $t \to +\infty$, for each $t > 0$.

Definition 2.1. *Let $(X, d), (Y, \rho)$ be metric spaces. An operator $f : X \to Y$ is said to be a φ-contraction if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and $\rho(f(x), f(y)) \leq \varphi(d(x, y))$, for all $x, y \in X$.*

For $f : X \to X$ we will denote $I_{b,cl}(f) := \{Y \in I_b(f) |\ \text{Y is closed}\ \}$ and $O_f(x) := \{x, f(x), f^2(x), \cdots\}$, for $x \in X$.

3. Main results

We start this section with the following known result.

Theorem 3.1. (J. Matkowski [5], I. A. Rus [7]) *Let (X, d) be a complete metric space and $f : X \to X$ a φ-contraction. Then $Fixf = \{x^*\}$ and $f^n(x_0) \to x^*$ when $n \to \infty$, for all $x_0 \in X$.*

Our first result is an extension of the above theorem to the case of a set X endowed with two metrics.

Theorem 3.2. *Let X be a nonempty set, and d, d' two metrics on X, Suppose that:*

i) (X, d) is a complete metric space;

ii) there exists $c > 0$ such that $d(x, y) \leq cd'(x, y)$ for each $x, y \in X$.

Let $f : X \to X$ be a φ-contraction with respect to d' and suppose that $f : (X, d) \to (X, d)$ is continuous.

Then

i) $Fixf = \{x^*\}$.

ii) If additionally, the mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ is continuous, strictly increasing and onto, then the fixed point problem for f is well posed with respect to d' (see [9]*), i.e., if $(x_n)_{n\in(N)}$ is a sequence in X such that $d'(x_n, f(x_n)) \to 0$ as $n \to \infty$, then $x_n \stackrel{d'}{\to} x^* \in Fixf$ as $n \to \infty$.*

Proof. i) **Step 1.** Let $x \in X$ be arbitrary. Then, we successively have:

$$d'(f^n(x), f^{n+1}(x)) \leq \varphi^n(d'(x, f(x))) \to 0 \text{ as } n \to \infty.$$

Hence $d'(f^n(x), f^{n+1}(x)) \to 0$ as $n \to \infty$, for each $x \in X$.
Step 2. We show now that the sequence $u_n := f^n(x)$ is Cauchy with respect to d'. For $\epsilon > 0$ choose $N \in \mathbb{N}$ such that $d'(u_n, u_{n+1}) < \delta(\epsilon) := \epsilon - \varphi(\epsilon)$, for all $n \in \mathbb{N}, n \geq N$.

If $d'(x, f(x)) < \delta(\epsilon)$ then for $z \in B_{d'}(x, \epsilon)$ we have $d'(f(z), x) \leq d'(f(z), f(x)) + d'(f(x), x) < \varphi(d'(z, x)) + \delta(\epsilon) \leq \varphi(\epsilon) + \epsilon - \varphi(\epsilon) = \epsilon$.
Hence $f(z) \in B_{d'}(x, \epsilon)$ and thus, we have proved that if $d'(x, f(x)) < \delta(\epsilon)$ then $f : B_{d'}(x, \epsilon) \to B_{d'}(x, \epsilon)$. Since $d'(u_N, f(u_N)) < \delta(\epsilon)$ we have that $f : B_{d'}(u_N, \epsilon) \to B_{d'}(u_N, \epsilon)$. Thus $u_{N+1} = f(u_N) \in B(u_N, \epsilon)$.

By induction we get that $u_{N+k} \in B(u_N, \epsilon)$, for all $k \in \mathbb{N}$. Then $d'(u_k, u_s) < d'(u_k, u_N) + d'(u_N, u_s) < 2\epsilon$, for all $k, s \in (N); k, s \geq N$. Therefore $(u_n)_{n\in(N)}$ is a Cauchy sequence with respect to d'. From b) it follows that $(u_n)_{n\in(N)}$ is Cauchy with respect to d too. Hence $u_n \stackrel{d}{\to} x^*, n \to \infty$.
Step 3. We will show that $x^* \in Fixf$. Since $u_n = f^n(x) \stackrel{d}{\to}$ as $n \to \infty$ by the continuity condition of f with respect to d we get that $x^* \in Fixf$.
Step 4. The uniqueness follows from the φ-contraction condition with respect to d'.
ii) Suppose $(x_n)_{n\in(N)}$ is a sequence in X such that $d'(x_n, f(x_n)) \to 0$ as $n \to \infty$. We have to prove that $x_n \stackrel{d'}{\to} x^* \in Fixf$. We have $d'(x_n, x^*) \leq d'(x_n, f(x_n)) + d'(f(x_n), x^*) \leq d'(x_n, f(x_n)) + \varphi(d'(x_n, x^*))$. Hence $\psi(d'(x_n, x^*)) \leq d'(x_n, f(x_n))$. Thus $d'(x_n, x^*) \leq \psi^{-1}(d'(x_n, f(x_n)))$. As $n \to \infty$, we get that $\lim\limits_{n\to\infty} d'(x_n, x^*) \leq \psi^{-1}(0) = 0$. Thus $\lim\limits_{n\to\infty} d'(x_n, x^*) = 0$. □

A local result of this type is:

Theorem 3.3. *Let X be a nonempty set, and d, d' two metrics on X, Suppose that*

i) (X, d) is a complete metric space,
ii) there exists $c > 0$ such that $d(x, y) \leq cd'(x, y)$ for each $x, y \in X$.

Let $x_0 \in X, r > 0$ and $f : \bar{B}^d_{d'}(x_0; r) \to X$ be a φ-contraction respect to d'. Suppose that $d'(x_0, f(x_0)) < r - \varphi(r)$ and $f : (X, d) \to (X, d)$ continuous.
Then:
A) $Fixf \cap \bar{B}^d_{d'}(x_0, r) = \{x^*\}$.
B) If additionally, we suppose that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ is continuous, strictly increasing and onto, then the fixed point problem for f is well posed with respect to d' metrics.

Proof. A) First we prove that $\bar{B}^d_{d'}(x_0, r) \in I(f)$ (i.e. $f : \bar{B}^d_{d'}(x_0, r) \to \bar{B}^d_{d'}(x_0, r)$)
Let $x \in \bar{B}^d_{d'}(x_0, r)$. Then, there exists $(x_n)_{n\in\mathbb{R}} \subset B_{d'}(x_0, r)$ such that $x_n \stackrel{d}{\to} x$. We

will show that $f(x) \in \bar{B}^d_{d'}(x_0, r)$, i.e., there exists $(y_n)_{n\in\mathbb{N}} \subset B_{d'}(x_0, r)$ such that $y_n \xrightarrow{d} f(x)$.

We have that $d'(f(x_n), x_0) \le d'(f(x_n), f(x_0)) + d'(f(x_0), x_0) < \varphi(d'(x_n, x_0)) + r - \varphi(r) < \varphi(r) + r - \varphi(r) = r$. Then $y_n := f(x_n) \in B_{d'}(x_0, r)$. Since $f : (X, d) \to (X, d)$ is continuous, letting $n \to \infty$ we have $y_n \xrightarrow{d} f(x)$. Thus $f : \bar{B}^d_{d'}(x_0, r) \to \bar{B}^d_{d'}(x_0, r))$ is φ- contraction with respect to metric d'. It follows that there exists a unique $x^* \in \bar{B}^d_{d'}(x_0, r) \cap Fixf$, by Theorem 3.2.

B) Let $(x_n) \in \bar{B}^d_{d'}(x_0, r)$ such that $d'(x_n, f(x_n)) \to 0$ as $n \to \infty$. By estimating $d'(x_n, x^*) \le d'(x_n, f(x_n)) + d'(f(x_n), x^*) = d'(x_n, f(x_n)) + d'(f(x_n), f(x^*)) \le d'(x_n, f(x_n)) + \varphi(d'(x_n, x^*)$. Then, $d'(x_n, x^*) - \varphi(d'(x_n, x^*) \le d'(x_n, f(x_n))$.

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ and suppose that is strictly increasing and onto. From $\psi(d'(x_n, x^*)) \le d'(x_n, f(x_n))$ we have $d'(x_n, x^*) \le \psi^{-1}(d'(x_n, f(x_n)))$ and because f is a comparison map and φ is continuous in 0, it follows that ψ^{-1} is continuous in 0. Letting $n \to \infty$ in the above relation $d'(x_n, x^*) \le \psi^{-1}(d'(x_n, f(x_n)))$, we obtain $\lim\limits_{n\to\infty} d'(x_n, x^*) \le \psi^{-1}(0) = 0$. Thus $d'(x_n, x^*) \to 0$ as $n \to \infty$. □

Consider now the case of the Caristi-type operators.

Theorem 3.4. *Let X be a nonempty set, and d, d' two metrics on X, Suppose that:*

i) (X, d) is a complete metric space;

ii) there exists $c > 0$ such that $d(x, y) \le cd'(x, y)$ for each $x, y \in X$.

Let $x_0 \in X, r > 0$ and $\varphi : X \to \mathbb{R}_+$ be a function such that $\varphi(x_0) < r$. Consider $f : \bar{B}^d_{d'}(x_0; r) \to X$ such that $d'(x, f(x)) \le \varphi(x) - \varphi(f(x))$, for each $x \in \bar{B}^d_{d'}(x_0; r)$. If f has closed graph with respect to the metric d or the function $x \mapsto d(x, f(x))$, $x \in \bar{B}^d_{d'}(x_0; r)$ is lower semicontinuous, then $Fixf \ne \emptyset$.

Proof. First we will show that the sequence $x_n := f^n(x_0), n \in \mathbb{N}^*$ is included in $\bar{B}^d_{d'}(x_0; r)$. Let $x_1 = f(x_0)$. Indeed, $d'(x_0, x_1) = d'(x_0, f(x_0)) \le \varphi(x_0) - \varphi(f(x_0)) = \varphi(x_0) - \varphi(x_1) \le \varphi(x_0) < r$. Hence $x_1 \in B_{d'}(x_0; r)$. Then $d'(x_1, x_2) = d'(x_1, f(x_1)) \le \varphi(x_1) - \varphi(f(x_1)) = \varphi(x_1) - \varphi(x_2)$ and $d'(x_0, x_2) \le d'(x_0, x_1) + d'(x_1, x_2) \le \varphi(x_0) - \varphi(x_1) + \varphi(x_1) - \varphi(x_2) = \varphi(x_0) - \varphi(x_2) < r$. Again we have $x_2 \in B_{d'}(x_0; r)$.

Inductively we obtain that:

(i) $x_n \in B_{d'}(x_0; r)$, for each $n \in \mathbb{N}^*$.

(ii) $d'(x_n, x_{n+1}) \le \varphi(x_n) - \varphi(x_{n+1})$, for each $n \in \mathbb{N}$.

From (ii) we have that $\sum\limits_{n=1}^{+\infty} d'(x_n, x_{n+1}) = d'(x_0, x_1) + d'(x_1, x_2) + ... \le \varphi(x_0) - \varphi(x_1) + \varphi(x_1) - \varphi(x_2) + ... \le \varphi(x_0)$, proving that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d'), and from the hypothesis ii) $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d) too. Denote by $x^* \in \bar{B}^d_{d'}(x_0; r)$ the limit of this sequence, i.e. $d(x_n, x^*) \to 0$.

If the graph of f is closed, since $x_{n+1} = f(x_n)$, we get that (x_n, x_{n+1}), converges to (x^*, x^*) and thus it follows that $(x^*, x^*) \in Graphf$. As a consequence, $x^* = f(x^*)$.

If the function $x \mapsto d(x, f(x))$, $x \in \bar{B}^d_{d'}(x_0; r)$ is lower semicontinuous, then $0 \le d(x^*, f(x^*)) \le \liminf\limits_{n\to+\infty} d(x_n, f(x_n)) = \liminf\limits_{n\to+\infty} d(x_n, x_{n+1}) = 0$, proving again that $x^* = f(x^*)$. □

The following three results are applications of the above local theorem of Caristi type.

Theorem 3.5. *Let X be a nonempty set, and d, d' two metrics on X. Suppose that:*

i) (X, d) *is a complete metric space;*
ii) *there exists* $c > 0$ *such that* $d(x, y) \leq cd'(x, y)$ *for each* $x, y \in X$*;*

Let $f : X \to X$*, be an operator and let* $x_0 \in X$*. Suppose that there exists* $a \in]0, 1[$ *such that* $d'(f(x), f^2(x)) \leq a \cdot d'(x, f(x))$*, for each* $x \in \bar{B}^d_{d'}(x_0; r)$ *and* $d'(x_0, f(x_0)) < (1-a)r$*. If* f *has a closed graph with respect to the metric* d *or the function* $x \mapsto d(x, f(x))$*,* $x \in \bar{B}^d_{d'}(x_0; r)$ *is lower semicontinuous, then* $Fixf \neq \emptyset$*.*

Proof. The condition imposed on f implies that

$$d'(x, f(x)) \leq \varphi(x) - \varphi(f(x)),\ \text{for each } x \in \bar{B}^d_{d'}(x_0; r),$$

where $\varphi(x) := \frac{1}{1-a} \cdot d'(x, f(x))$. Moreover $\varphi(x_0) = \frac{1}{1-a} \cdot d'(x_0, f(x_0)) < r$. The conclusion follows from Theorem 3.4. □

Another consequence of Caristi's theorem for operators defined on a ball will be considered now. In this respect, we need a definition.

Definition 3.1. Let (X, d) be a metric space and $f : X \to X$ an operator. Denote $O_f(x, y) := O_f(x) \cup O_f(y)$. Then, by definition, an element $x \in X$ is said to be regular if its orbit is bounded, i. e. $diamO_f(x) < +\infty$.

Then we have:

Theorem 3.6. *Let X be a nonempty set, and d, d' two metrics on X, Suppose that:*

i) (X, d) *is a complete metric space;*
ii) *there exists* $c > 0$ *such that* $d(x, y) \leq cd'(x, y)$ *for each* $x, y \in X$.

Let $f : X \to X$ *be an operator with bounded orbits. Suppose that there exists* $a \in [0, 1[$ *such that* $diamO^{d'}_f(f(x)) \leq a \cdot diamO^{d'}_f(x)$*, for each* $x \in X$*. If* f *has closed graph with respect to the metric* d *or the function* $x \mapsto diamO^d_f(x)$ *is lower semicontinuous, then* $Fixf \neq \emptyset$*.*

Proof. Consider $\varphi : X \to \mathbb{R}_+$, defined by $\varphi(x) := \frac{1}{1-a} \cdot diamO^{d'}_f(x)$. Then the conclusion follows now from the global result corresponding to Theorem 3.4, see [2] and [3]. □

Next result is again a local one:

Theorem 3.7. *Let X be a nonempty set, and d, d' two metrics on X, Suppose that:*

i) (X, d) *is a complete metric space;*
ii) *there exists* $c > 0$ *such that* $d(x, y) \leq cd'(x, y)$ *for each* $x, y \in X$.

Let $x_0 \in X$ *and* $r > 0$*,* $f : X \to X$ *be an operator with bounded orbits. Suppose that there exists* $a \in [0, 1[$ *such that* $diamO^{d'}_f(f(x)) \leq a \cdot diamO^{d'}_f(x)$*, for each* $x \in \bar{B}^d_{d'}(x_0; r) \cap O_f(x_0)$ *and* $diamO^{d'}_f(x_0) < (1-a)r$*. If* f *has closed graph with respect to the metric* d *or the function* $x \mapsto d(x, f(x))$*,* $x \in \bar{B}^d_{d'}(x_0; r)$ *is lower semicontinuous, then* $Fixf \neq \emptyset$*.*

Proof. Consider the function $\varphi : X \to \mathbb{R}_+$, defined by $\varphi(x) := \frac{1}{1-a} \cdot diamO_f^{d'}(x)$. The conclusion follows from Theorem 3.4. □

Remark 3.1. Some applications of the above results to domain invariance principles will be given in a forthcoming paper.

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