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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} anniversary

Dynamic transport problems of cost type and time type

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ABSTRACT. Real problems which arise in Health Economy are mathematically modeled by means of dynamic optimization problems of transport type. The aim of this paper is to study them and to give some methods to solve them.

1. INTRODUCTION

The transport problem has the largest applicability among all the other optimization problems. Its study had generated the duality in the linear programming and, consequently, the method (or technique) of potential plans has been developed. Starting with the solving of a problem of planning the mammography in the frame of a screening program for the mammal cancer, in [1] a particular type of dynamic optimization problem, named Lexicographic Bi-criteria Dynamic Transport Problem of Cost-Time Type, has been obtained. Also, another particular type of lexicographical dynamic optimization problem, applied in cervical cancer screening, is presented in [2].

In the domain of Health Economy, there exist many other problems whose mathematical model is, in essence, an optimization problem of transport type. For instance, the persons who suffer from a chronic disease are obliged to make a special analysis which can be done only in specialized places (laboratories). Given the home addresses of the patients, the addresses of the laboratories, the cost transport between the towns (if one laboratory is in another city than the city where the patient lives) and the number of days when all the patients have to make the analysis, we get a problem of planning this activity according to the days, such as the total cost for the transport to be minimum (supposing that the cost is supported by the screening program). In the case when the patient pays for the travel, it is important to plan (according to the days) this activity such as the greatest cost that he (or she) pays to be the lowest possible one. These two situations generate two different types of optimization problems. As far as we know, these types of transport problems have not been studied in literature.

In both cases we consider the problem as a dynamic discrete and finite process, with the number of states equal to the number of days in which the whole

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	L_1	L_2	L_3
First day	1	4	7
Second day	10	4	8
Thirty day	10	10	10

TABLE 1. Capacity of labs

	L_1	L_2	L_3
S_1	1	3	7
S_2	8	5	2

TABLE 2. Transport costs

activity takes place. The static and dynamic equations will be the same, the difference being only in the computation of the economical effect on the stages and the total economical effect. In our paper, we give a method for solving each of these problems, indicating by means of some examples, the necessity of studying such problems.

Example 1.1. Let's suppose that in the city S_1 there exist 20 patients and in the city S_2, 15 patients. During 3 days they have to make an analysis at one of the three laboratories L_1, L_2 or L_3. The maximum number of analysis that can be make in the 3 days at every laboratory is given in Table 1.

The cost between cities, given in Table 2, is also known.

We want to plan the patients at the 3 laboratories such that in the 3 days, all the analysis to be made and the transport cost to be minimum. Denoting by x_{ij}^k the number of patients from the city S_i, $i \in \{1,2\}$, which in the day k, $k \in \{1,2,3\}$, make the analysis at lab L_j, $j \in \{1,2,3\}$, the following problem has to be solved:

$$\begin{cases} \sum\limits_{k=1}^{3}(x_{11}^k + 3x_{12}^k + 7x_{13}^k + 8x_{21}^k + 5_{22}^k + 2x_{23}^k) \;\to\; \min \\ \sum\limits_{k=1}^{3}(x_{11}^k + x_{12}^k + x_{13}^k) \;=\; 20 \\ \sum\limits_{k=1}^{3}(x_{21}^k + x_{22}^k + x_{23}^k) \;=\; 15 \\ x_{11}^1 + x_{21}^1 \;\le\; 1 \\ x_{12}^1 + x_{22}^1 \;\le\; 4 \\ x_{13}^1 + x_{23}^1 \;\le\; 7 \\ x_{11}^2 + x_{21}^2 \;\le\; 10 \\ x_{12}^2 + x_{22}^2 \;\le\; 4 \\ x_{13}^2 + x_{23}^2 \;\le\; 8 \\ x_{11}^3 + x_{21}^3 \;\le\; 10 \\ x_{12}^3 + x_{22}^3 \;\le\; 10 \\ x_{13}^3 + x_{23}^3 \;\le\; 10 \\ x_{ij}^k \in \mathbf{N}, \text{ for all } i \in \{1,2\},\; j \in \{1,2,3\},\; k \in \{1,2,3\}. \end{cases}$$

Relaxing this problem by substituting the request $x_{ij}^k \in \mathbf{N}$ by $x_{ij}^k \geq 0$ for all $i \in \{1,2\}$, $j \in \{1,2,3\}$, $k \in \{1,2,3\}$, we get a linear optimization problem which can be solved by means of the simplex algorithm. In addition, in case of existence, the obtained optimal solution, being a vertex of the admissible solutions polyhedral, will satisfy also the integrity condition, which means that will be the optimal solution of our initial problem.

Because in practice the number of variables is great and the restrictions of the problem are similar to those of a transport problem, we give a specific method for solving it. It is based on two algorithms: one, which uses the potential plans technique, and another, generated by approaching our problem as a dynamic process with a finite number of stages.

2. Dynamic transport problems of cost type and of time type

Let m, n, p, a_i, $i \in \{1,...,m\}$ and b_j^k, $j \in \{1,...,n\}$, $k \in \{1,...,p\}$ be natural (non null) numbers, and let c_{ij}, t_{ij}, $i \in \{1,...,m\}$, $j \in \{1,...,n\}$, be non negative real numbers.

Let us denote

$$I = \{1,...,m\}, \ J = \{1,...,n\}, \ K = \{1,...,p\},$$

$$S_0 = \{(a_1,...,a_m)\}, \tag{2.1}$$

and

$$S_p = \{(0,...,0)\}. \tag{2.2}$$

For each $k \in \{1,...,p-1\}$, let us set

$$S_k := \{0,1,...,a_1\} \times ... \times \{0,1,...,a_m\}. \tag{2.3}$$

Also, if $k \in \{1,...,p-1\}$, and $s^{k-1} \in S_{k-1}$, by $\Lambda_k(s^{k-1})$, we denote the set of the solutions of the system

$$\sum_{i=1}^{m} x_{ij}^k \leq b_j^k, \text{ for all } j \in \{1,...,n\}, \tag{2.4}$$

$$\sum_{j=1}^{n} x_{ij}^k \leq s_i^{k-1}, \text{ for all } i \in \{1,...,m\}, \tag{2.5}$$

$$x_{ij}^k \in \mathbb{N}, \text{ for all } i \in \{1,...,m\}, \ j \in \{1,...,n\}. \tag{2.6}$$

If $k = p$, and $s^{p-1} \in S_{p-1}$, we denote by $\Lambda_p(s^{p-1})$, the set of the solutions of the system

$$\sum_{i=1}^{m} x_{ij}^p \leq b_j^p, \text{ for all } j \in \{1,...,n\}, \tag{2.7}$$

$$\sum_{j=1}^{n} x_{ij}^p = s_i^{p-1}, \text{ for all } i \in \{1,...,m\}, \tag{2.8}$$

$$x_{ij}^p \in \mathbb{N}, \text{ for all } i \in \{1,...,m\}, \ j \in \{1,...,n\}. \tag{2.9}$$

A *dynamic transport problem* is a discrete finite stages decision problem, its dynamic equations and its static equations having the following expressions:

$$s_i^k = s_i^{k-1} - \sum_{j=1}^{n} x_{ij}^k, \text{ for all } i \in \{1,...,m\}, \text{ and } k \in \{1,...,p\}, \tag{2.10}$$

$$s^k \in S_k, \text{ for all } k \in \{0,1,...,p\}, \tag{2.11}$$

$$X^k = [x_{ij}^k] \in \Lambda(s^{k-1}), \text{ for all } k \in \{1,...,p\}. \tag{2.12}$$

S_k, the set of all states of the system in the stage $k \in \{0,1,...,p\}$, is given by (2.1), (2.2), (2.3). The set $\Lambda(s^{k-1})$ denotes the set of all decisions which may be taken in the stage k, if the system is in the state s^{k-1}; it is equal to the set of the solutions of the system (2.4)-(2.6), if $k \in \{1,...,p-1\}$, and to the set of the solutions of the system (2.7)-(2.9), if $k = p$.

A sequence $(X^1,...,X^p)$, where $X^k \in X_k(s^{k-1})$, for every $k \in \{1,...,p\}$, is called a *policy of the dynamic transport problem*. The set of all the policies of the dynamic transport problem will be denoted by Pol.

We will say that *a dynamic transport problem is of the cost type* and we denote such problem by (DTC), if

C1. at each stage $k \in \{1,...,p\}$, if we take the decision $X^k = [x_{ij}^k] \in \Lambda_k(s^{k-1})$, the obtained utility, denoted by $f_k(X^k)$, is given by

$$f_k^C(X^k) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} \cdot x_{ij}^k. \tag{2.13}$$

C2. the total utility function $F : Pol \to \mathbb{R}$ is given by

$$F^C(X^1,...,X^p) = \sum_{k=1}^{p} f_k(X^k) = \sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} \cdot x_{ij}^k. \tag{2.14}$$

We say that *a dynamic transport problem is of the time type* and we denote such a problem by (DTT), if

T1. at each stage $k \in \{1,...,p\}$, if we take the decision $X^k = [x_{ij}^k] \in \Lambda_k(s^{k-1})$, the obtained utility, denoted by $f_k(X^k)$, is given by

$$f_k^T(X^k) = \max\{t_{ij} \cdot \text{sign}(x_{ij}^k) \,|\, i \in I,\ j \in J\}. \tag{2.15}$$

T2. the total utility function $F : Pol \to \mathbb{R}$ is given by

$$F^T(X^1,...,X^p) = \max\{t_{ij} \cdot \text{sign}(x_{ij}^k) \,|\, i \in I,\ j \in J,\ k \in K\}. \tag{2.16}$$

3. Global problems attached to the problems (DTC) and (DTT)

Let

$$b_j^* := \sum_{k=1}^{p} b_j^k, \text{ for all } j \in \{1,...,n\}, \tag{3.17}$$

We consider the system

$$\begin{cases} \sum\limits_{i=1}^{m} w_{ij} \ \le \ b_j^*, \ \text{ for all } \ j \in \{1, ..., n\}, \\ \sum\limits_{j=1}^{n} w_{ij} \ = \ a_i, \ \text{ for all } \ i \in \{1, ..., m\}, \\ w_{ij} \in \mathbf{N}, \ \text{ for all } \ i \in \{1, ..., m\}, \ j \in \{1, ..., n\}, \end{cases} \tag{3.18}$$

A solution of this system will be denoted by $W = [w_{ij}]$ and the set of all its solutions by $\mathbf{W}$.

Let the functions $FC : \mathbf{W} \to \mathbf{R}$, $FC(W) = \sum\limits_{i \in I} \sum\limits_{j \in J} c_{ij} w_{ij}$, for all $W \in \mathbf{W}$, and $FT : \mathbf{W} \to \mathbf{R}$, $FT(W) = \max\{t_{ij} \cdot \text{sign} w_{ij} \mid i \in I, \ j \in J\}$ for all $W \in \mathbf{W}$.

We attach to Problem (DTC), respective to Problem (DTT), the following classical transport problem of cost type, respective of time type.

$$(GC) \quad \begin{cases} FC(W) \to \min \\ W \in \mathbf{W} \end{cases} \qquad\qquad (GT) \quad \begin{cases} FT(W) \to \min \\ W \in \mathbf{W} \end{cases}$$

We mention that the problems (GC) and (GT) can be solved using, for example, the corresponding algorithms given in [3].

Proposition 3.1. *If $(X^1, ..., X^p)$ is a policy of a dynamic transport problem (DTC) or (DTT), then the matrix $W = [w_{ij}]$, where*

$$w_{ij} \ = \ \sum_{k \in K} x_{ij}^k, \ \textit{ for all } i \in I, \ j \in J, \tag{3.19}$$

is a solution of the system (3.18),

$$F^C(X^1, ..., X^p) \ = \ FC(W) \tag{3.20}$$

and

$$F^T(X^1, ..., X^p) \ = \ FT(W). \tag{3.21}$$

Proof. As $(X^1, ..., X^p) \in Pol$, in view of (2.4), (2.7) and (3.17), we have

$$\sum_{i \in I} w_{ij} \ = \ \sum_{k \in K} \sum_{i \in I} x_{ij}^k \ \le \ \sum_{k \in K} b_j^k \ = \ \beta_j, \ \text{ for all } j \in J. \tag{3.22}$$

Let $i \in I$. From (2.8) we get that

$$\sum_{k \in K} \sum_{j \in J} x_{ij}^k \ = \ \sum_{k=1}^{p-1} \sum_{j \in J} x_{ij}^k \ + \ s_i^{p-1}.$$

But, from (2.10), we obtain

$$s_i^{p-1} \ = \ s_i^0 \ - \ \sum_{k=1}^{p-1} \sum_{j \in J} x_{ij}^k.$$

Therefore

$$\sum_{k \in K} \sum_{j \in J} x_{ij}^k \ = \ \sum_{k=1}^{p-1} \sum_{j \in J} x_{ij}^k \ + \ s_i^0 \ - \ \sum_{k=1}^{p-1} \sum_{j \in J} x_{ij}^k \ = \ s_i^0.$$

But from (2.1), $s_i^0 = a_i$. This implies that, for each $i \in I$, we have

$$\sum_{j\in J} w_{ij} = \sum_{k=1}^{p}\left(\sum_{j\in J} x_{ij}^k\right) = a_i. \tag{3.23}$$

As $x_{ij}^k \in \mathbf{N}$, for all $i \in I$, $j \in J$,

$$w_{ij} = \sum_{k\in k} x_{ij}^k \in \mathbf{N}. \tag{3.24}$$

Therefore, $W = [w_{ij}]$ is a feasible solution.

In addition, from (2.14), we have

$$FC(W) = \sum_{i\in I}\sum_{j\in J} c_{ij}w_{ij} = \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} c_{ij}x_{ij}^k = F^C(X^1, ..., X^p).$$

As

$$t_{ij}\cdot \text{sign}(w_{ij}) = \max\{t_{ij}\cdot \text{sign}(x_{ij}^k) \,|\, k \in K\},$$

from (2.16) we get

$$\begin{aligned} FT(W) = & \max\{t_{ij}\cdot \text{sign}(w_{ij}) \,|\, i\in I,\ j \in J\} = \\ = & \max\{t_{ij}\cdot \text{sign}(x_{ij}^k) \,|\, i\in I,\ j\in J,\ k\in K\} = F^T(X^1, ..., X^p). \end{aligned}$$

□

Proposition 3.2. *If $W = [w_{ij}]$ is a feasible solution of the problem (GC), then there is a policy $(X^1, ..., X^p)$ of the dynamic transport problem (DTC) such that (3.19) and (3.20) hold.*

Proof. The proof is based on the following algorithm, named Dynamic Distribution Algorithm (DDA).

Step 1. Set $\beta_{0j}^k := b_j^k$, for all $k \in K$, $j \in J$, and $z_{ij}^0 := w_{ij}$, for all $i \in I$, $j \in J$.
Step 2. Set $i := 1$;
Step 3. Set $j := 1$;
Step 4. Set $k := 1$;
Step 5. Set $x_{ij}^k := \min\{\beta_{i-1,j}^k,\ z_{ij}^{k-1}\}$;
Step 6. Set $\beta_{ij}^k := \beta_{i-1,j}^k - x_{ij}^k$;
Step 7. Set $z_{ij}^k := z_{ij}^{k-1} - x_{ij}^k$;
Step 8. Set $k := k+1$;
Step 9. If $k \le p$, then go back to Step 5; else proceed;
Step 10. Set $j := j+1$;
Step 11. If $j \le n$, then go back to Step 4; else proceed;
Step 12. Set $i := i+1$;
Step 13. If $i \le m$, then go back to Step 3; else proceed;
Step 14. Stop.

We have to prove that $(X^1, ..., X^p)$, where $X^k = [x_{ij}^k]$ for all $k \in K$, is a policy of (DTC). As $b_j^k \in \mathbf{N}$, for all $j \in J$, $k \in K$, from steps 1, 5 and 6 we get that $0 \le \beta_{ij}^k \le b_j^k$ and $\beta_{ij}^k \in \mathbf{N}$, for every $i \in I$, $j \in J$, $k \in K$. Then, from steps 1, 5, and 7 we deduce that, for every $i \in I$, $j \in J$, $k \in K$ we have $x_{ij}^k \in \mathbf{N}$.

	L_1	L_2	L_3	
S_1	1	3	7	20
S_2	8	5	2	15
S_3	0	0	0	29
	21	18	25	64

TABLE 3. The table corresponding to the problem (GC)

From step 6 we obtain

$$\sum_{i=1}^{m} x_{ij}^k = \sum_{i=1}^{m} (\beta_{i-1,j}^k - \beta_{ij}^k) = \beta_{0j}^k - \beta_{mj}^k \leq \beta_{0j}^k = b_j^k.$$

Let $i \in I$. From steps 5, 6, 7 and 1 we get that $\sum_{k\in K} x_{ij}^k = z_{ij}^0 = w_{ij}$. This implies that $\sum_{k\in K}\sum_{j\in J} x_{ij}^k = \sum_{j\in J} w_{ij} = a_i$, because $\sum_{j\in J} w_{ij} = a_i$. Then

$$F^C(X^1, .., X^p) = \sum_{k\in K}\sum_{j\in J}\sum_{i\in I} c_{ij}\cdot x_{ij}^k = \sum_{j\in J}\sum_{i\in I} c_{ij}\cdot w_{ij} = FC(W).$$

□

It is very easy to see that

Corollary 3.1. *If $W = [w_{ij}]$ is an optimal solution of the problem (GC), then the corresponding policy $(X^1, ..., X^p)$ given by the DDA Algorithm is an optimal policy of the problem (DTC).*

Therefore, if we have to solve the problem (DTC), first we will solve the corresponding (GC). Then we will apply the DDA Algorithm. The policy $(X^1, ..., X^p)$ given by this algorithm is, in view of Corollary 3.1, an optimal policy of the problem (DTC).

Example 3.2. If we consider the application given in Example 1.1, first we will solve the classical transport problem of cost type given in the Table 3 where S_3 is a fictive locality. It is introduced in order to transform the initial (GC) problem in a well-balanced transport problem. An optimal solution of the initial (GC) problem is the matrix

$$\begin{pmatrix} 20 & 0 & 0 \\ 0 & 0 & 15 \end{pmatrix}.$$

Now, if we apply DDA Algorithm, we obtain successive: $x_{11}^1 := 1$, $x_{11}^2 := 10$, $x_{11}^3 := 9$, $x_{12}^1 := 0$, $x_{12}^2 := 0$, $x_{12}^3 := 0$, $x_{13}^1 := 0$, $x_{13}^2 := 0$, $x_{13}^3 := 0$, $x_{21}^1 := 0$, $x_{21}^2 := 0$, $x_{21}^3 := 0$, $x_{22}^1 := 0$, $x_{22}^2 := 0$, $x_{22}^3 := 0$, $x_{23}^1 := 7$, $x_{23}^2 := 8$, $x_{23}^3 := 0$. Therefore $(X^1 = [x_{ij}^1], X^2 = [x_{ij}^2], X^3 = [x_{ij}^3])$, is an optimal policy of the our practical problem.

Proposition 3.3. *If $W = [w_{ij}]$ is a feasible solution of the problem (GT), then there is a policy $(X^1, ..., X^p)$ of the dynamic transport problem (DTT) such that (3.19) holds and $F^T(X^1, ..., X^p) = FT(W)$.*

Proof. Applying the DDA Algorithm, we obtain the policy $(X^1, ..., X^p)$ which satisfies (3.19). As

$$\text{sign}(\sum_{k\in K} x_{ij}^k) = \max\{\text{sign}(x_{ij}^k) \mid k \in K\}$$

we get that

$$\begin{aligned} F^T(X^1, ..., X^p) = \quad & \max\{t_{ij} \cdot \text{sign}(x_{ij}^k) \mid i \in I,\ j \in J,\ k \in K\} = \\ & \max\{t_{ij} \cdot \text{sign}(\sum_{k\in K} x_{ij}^k) \mid i \in I,\ j \in J\} = \\ & FT(W). \end{aligned}$$

$\square$

Then, the following result holds:

Corollary 3.2. *If $W = [w_{ij}]$ is an optimal solution of the problem (GT), then the corresponding policy $(X^1, ..., X^p)$ given by DDA Algorithm is an optimal policy of the problem (DTT).*

Therefore, if we have to solve the problem (DTT), first we will solve the corresponding (GT) problem. Then we will apply the algorithm 1. In view of Corollary 3.2, the policy $(X^1, ..., X^p)$ given by this algorithm is an optimal policy of the problem (DTT).

Remark 3.1. *DDA Algorithm gives, also, the minimum number of days necessary for the patients to do the analysis.*

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