CREATIVE MATH. & INF. **Online version at** `http://creative-mathematics.ubm.ro/`
17 (2008), No. 3, 466 - 472 **Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X**

Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} anniversary

Gamma approximating operators

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ABSTRACT. By using the gamma distribution we shall define a general linear transform $\Gamma^{(a)}_{\alpha,\beta}f$, $a \in \mathbb{R}$, from which we obtain as special cases the gamma first kind transform. For different value of α, β and a we obtain several gamma type operators studied in literature. We apply the gamma first kind transform to Szász-Mirakjan's operator and we obtain the Jain-Pethe operator.

1. INTRODUCTION

In this paper we continue our earlier investigations [4], [5], [6] concerning to use Euler's gamma distribution for constructing linear positive operators.

In probability theory and statistics, the gamma distribution is a two parameter family of continuous probability distributions. It has the shape parameter k and a scale parameter θ.

The probability density function of the gamma distribution can be expressed in terms of the gamma function parametrized in terms of a shape parameter k and scale parameter θ:

$$f(t;k,\theta) = \frac{1}{\theta^k \Gamma(k)} t^{k-1} e^{-x/\theta} \quad \text{for} \quad t > 0 \quad \text{and} \quad k,\theta > 0. \tag{1.1}$$

The Erlang distribution is a special case of the gamma distribution, where the shape parameter k is an integer. This distribution is sometimes called the Erlang-k distribution.

Alternatively, the gamma distribution can be parametrized in terms of a shape parameter $\alpha = k$ and an inverse scale parameter $\beta = 1/\theta$, called a rate parameter

$$g(t;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} \quad \text{for} \quad t > 0 \quad \text{and} \quad \alpha,\beta > 0. \tag{1.2}$$

By using the gamma distribution we shall define a general linear transform $\Gamma^{(a)}_{\alpha,\beta}f$, $a \in \mathbb{R}$, from which we obtain as special cases the gamma first kind transform. For different value of α, β and a we obtain several gamma type operators studied in literature. We apply the gamma first kind transform to Szász-Mirakjan's operator and we obtain the Jain-Pethe operator.

Received: 19.02.2008. In revised form: 10.10.2008. Accepted: 23.05.2009.
2000 *Mathematics Subject Classification*. 41A36.
Key words and phrases. *Euler's gamma distribution, gamma transform, positive linear operators.*

2. The gamma transform

By using (1.2) we define the gamma transform of a function f

$$\Gamma^{(a)}_{\alpha,\beta} f = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\beta t} f(t^a) dt \tag{2.1}$$

where $a \in \mathbb{R}$ and $f \in L_{1,loc}(0,\infty)$ such that $\Gamma^{(a)}_{\alpha,\beta}|f| < \infty$.

One observes that $\Gamma^{(a)}_{\alpha,\beta}$ is a positive linear functional.

Theorem 2.1. *The moment of order k of the functional $\Gamma^{(a)}_{\alpha,\beta}$ has the following value*

$$\Gamma^{(a)}_{\alpha,\beta} e_k = \frac{1}{\beta^{ak}} \cdot \frac{\Gamma(\alpha + ak)}{\Gamma(\alpha)}. \tag{2.2}$$

Proof. By using (2.1) we obtain

$$\Gamma^{(a)}_{\alpha,\beta} e_k = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\beta t} (t^a)^k dt = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha+ak-1} e^{-\beta t} dt$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+ak)}{\beta^{\alpha+ak}} = \frac{1}{\beta^{ak}} \cdot \frac{\Gamma(\alpha+ak)}{\Gamma(\alpha)}.$$

□

Consequently we obtain

$$\Gamma^{(a)}_{\alpha,\beta} e_1 = \frac{\Gamma(\alpha+a)}{\beta^a \Gamma(\alpha)}, \quad \Gamma^{(a)}_{\alpha,\beta} e_2 = \frac{\Gamma(\alpha+2a)}{\beta^{2a} \Gamma(\alpha)}. \tag{2.3}$$

We impose $\Gamma^{(a)}_{\alpha,\beta} e_1 = e_1$, that is $x = \dfrac{\Gamma(\alpha+a)}{\beta^a \Gamma(\alpha)}$ and we obtain

$$\Gamma^{(a)}_{\alpha,\beta}((t-x)^2; x) = \frac{\Gamma(\alpha)\Gamma(\alpha+2a) - \Gamma^2(\alpha+a)}{\Gamma^2(\alpha+a)} x^2.$$

We shall consider here the special cases $a = 1$ and $a = -1$.

3. Special cases

3.1. **The gamma transform. Case $a = 1$.** If we consider $a = 1$ in (2.1) we obtain the gamma transform of a function f

$$\Gamma_{\alpha,\beta} f = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\beta t} f(t) dt \tag{3.1}$$

where $f \in L_{1,loc}(0,\infty)$ such that $\Gamma_{\alpha,\beta}|f| < \infty$.

Corollary 3.1. *The moment of order k of the functional $\Gamma_{\alpha,\beta}$ has the following value*

$$\Gamma_{\alpha,\beta} e_k = \frac{1}{\beta^k} \cdot \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \frac{(\alpha)_k}{\beta^k}.$$

Proof. The result follows from Theorem 2.1 for $a = 1$. □

Consequently we obtain

$$\Gamma_{\alpha,\beta}e_1 = \frac{\alpha}{\beta}, \quad \Gamma_{\alpha,\beta}e_2 = \frac{\alpha(\alpha+1)}{\beta^2}. \tag{3.2}$$

We impose $\Gamma_{\alpha,\beta}e_1 = e_1$, that is $\frac{\alpha}{\beta} = x$, or $\beta = \frac{\alpha}{x}$ and we obtain the following linear positive operators

$$(\Gamma_\alpha f)(x) = \left(\frac{\alpha}{x}\right)^\alpha \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\frac{\alpha}{x}t} f(t)dt \tag{3.3}$$

or, equivalent

$$(\Gamma_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} e^{-u} f\left(\frac{ux}{\alpha}\right) du.$$

By (3.2) we obtain

$$\Gamma_\alpha((t-x)^2; x) = \frac{x^2}{\alpha}. \tag{3.4}$$

If we choose $\alpha = n$, $n \in \mathbb{N}$ in (3.3) then we obtain Post-Wider's positive linear operator, defined for $f \in L_{1,loc}(0,\infty)$ by

$$(P_n f)(x) = \frac{1}{\Gamma(n)} \left(\frac{n}{x}\right)^n \int_0^\infty t^{n-1} e^{-\frac{nt}{x}} f(t)dt, \tag{3.5}$$

If we replace $\alpha = nx$, $n \in \mathbb{N}$ in (3.3), we obtain Rathore's positive linear operator, defined for $f \in L_{1,loc}(0,\infty)$ by

$$(R_n f)(x) = \frac{n^{nx}}{\Gamma(nx)} \int_0^\infty t^{nx-1} e^{-nt} f(t)dt \tag{3.6}$$

Corollary 3.2. *One has*

$$P_n((t-x)^2; x) = \frac{x^2}{n}, \quad R_n((t-x)^2; x) = \frac{x}{n}.$$

Proof. It is obtained from (3.4) for $\alpha = n$ and respectively $\alpha = nx$. □

3.2. **The gamma transform. Case** $a = -1$**.** We consider now the case $a = -1$. If we put $a = -1$ in (2.1) we obtain the gamma transform of a function f

$$\widetilde{\Gamma}_{\alpha,\beta} f = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\beta t} f\left(\frac{1}{t}\right) dt \tag{3.7}$$

where $f \in L_{1,loc}(0,\infty)$ such that $\Gamma_{\alpha,\beta}|f| < \infty$.

Corollary 3.3. *The moment of order k of the functional $\Gamma_{\alpha,\beta}$ has the following value*

$$\widetilde{\Gamma}_{\alpha,\beta} e_k = \frac{\beta^k \Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad k < \alpha.$$

Proof. The result follows from Theorem 2.1 for $a = -1$. □

Consequently we obtain

$$\widetilde{\Gamma}_{\alpha,\beta} e_1 = \frac{\beta}{\alpha-1}, \quad \widetilde{\Gamma}_{\alpha,\beta} e_2 = \frac{\beta^2}{(\alpha-1)(\alpha-2)}. \tag{3.8}$$

We impose $\widetilde{\Gamma}_{\alpha,\beta}e_1 = e_1$ that is $\dfrac{\beta}{\alpha-1} = x$, or $\beta = (\alpha-1)x$ and we obtain the following linear positive operators

$$(\widetilde{\Gamma}_\alpha f)(x) = \frac{(\alpha-1)^\alpha x^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-(\alpha-1)tx} f\left(\frac{1}{t}\right) dt \tag{3.9}$$

By (3.8) we obtain

$$\widetilde{\Gamma}_\alpha((t-x)^2; x) = \frac{x^2}{\alpha-2}, \quad \alpha > 2. \tag{3.10}$$

For $\alpha = n+1$, $n \in \mathbb{N}$ we obtain the operator

$$(G_n f)(x) = \frac{(nx)^{n+1}}{\Gamma(n+1)} \int_0^\infty t^n e^{-ntx} f\left(\frac{1}{t}\right) dt \tag{3.11}$$

which is the gamma operator introduced and studied by A. Lupaş and M. Müller.

4. The functional $M_n^{(\alpha)} = \Gamma_\alpha(S_n f)$

It is well-known the operator of Szász-Mirakjan defined by

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^\infty \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{4.1}$$

where f is any function defined on $[0,\infty)$ such that $(S_n|f|)(x) < \infty$.

The operator S_n was generalized by Pethe and Jain [2] and studied by Miheşan [7], obtaining $M_n^{(\beta)}$ operator

$$(M_n^{(\beta)} f)(x) = \tag{4.2}$$
$$(1+n\beta)^{-x/\beta} \sum_{k=0}^\infty \left(\beta + \frac{1}{n}\right)^{-k} \frac{x(x+\beta)\dots(x+(k-1)\beta)}{k!} f\left(\frac{k}{n}\right)$$

where β is a nonnegative parameter depending on the natural number n and f is any real function defined on $[0,\infty)$ with $(M_n^{(\beta)}|f|)(x) < \infty$.

We obtain this operator (in equivalent form) if we apply gamma transform (3.3) to Szász operators (4.1).

Theorem 4.2. *The Γ_α transform of $S_n f$ can be expressed by the following form*

$$(M_n^{(\alpha)} f)(x) = \Gamma_\alpha(S_n f)(x) = \sum_{k=0}^\infty m_{n,k}(x,\alpha) f\left(\frac{k}{n}\right) \tag{4.3}$$

with

$$m_{n,k}(x,\alpha) = \frac{(\alpha)_k}{k!} \cdot \frac{\left(\dfrac{nx}{\alpha}\right)^k}{\left(1 + \dfrac{nx}{\alpha}\right)^{\alpha+k}} \tag{4.4}$$

where $x \geq 0$, $\alpha \in \mathbb{R}$, $\alpha + nx > 0$.

Proof.

$$(M_n^{(\alpha)}f)(x) = \Gamma_\alpha(S_nf)(x) = \left(\frac{\alpha}{x}\right)^\alpha \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-\frac{\alpha t}{x}}(S_nf)(t)dt$$

$$= \left(\frac{\alpha}{x}\right)^\alpha \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-\frac{\alpha t}{x}}e^{-nt}\sum_{k=0}^\infty \frac{(nt)^k}{k!}f\left(\frac{k}{n}\right)dt$$

$$= \left(\frac{\alpha}{x}\right)^\alpha \frac{1}{\Gamma(\alpha)}\sum_{k=0}^\infty \frac{n^k}{k!}f\left(\frac{k}{n}\right)\int_0^\infty t^{\alpha+k-1}e^{-\left(\frac{\alpha}{x}+n\right)t}dt$$

$$= \left(\frac{\alpha}{x}\right)^\alpha \frac{1}{\Gamma(\alpha)}\sum_{k=0}^\infty \frac{n^k}{k!}f\left(\frac{k}{n}\right)\frac{\Gamma(\alpha+k)}{\left(\frac{\alpha}{x}+n\right)^{\alpha+k}}$$

$$= \left(\frac{\alpha}{x}\right)^\alpha \sum_{k=0}^\infty \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\cdot\frac{n^k}{k!}\cdot\frac{x^{\alpha+k}}{(\alpha+nx)^{\alpha+k}}f\left(\frac{k}{n}\right)$$

$$= \alpha^\alpha \sum_{k=0}^\infty \frac{(\alpha)_k}{k!}\cdot\frac{(nx)^k}{(\alpha+nx)^{\alpha+k}}f\left(\frac{k}{n}\right) = \sum_{k=0}^\infty \frac{(\alpha)_k}{k!}\cdot\frac{\left(\frac{nx}{\alpha}\right)^k}{\left(1+\frac{nx}{\alpha}\right)^{\alpha+k}}f\left(\frac{k}{n}\right)$$

□

Lemma 4.1. *For $x \geq 0$, $n = 1, 2, \ldots$, we have*

$$M_n^{(\alpha)}e_0 = e_0, \quad M_n^{(\alpha)}e_1 = e_1, \quad (M_n^{(\alpha)}e_2)(x) = x^2 + \frac{x(nx+\alpha)}{n\alpha}.$$

Proof. The result follows from (3.4) and

$$S_ne_0 = e_0, \quad S_ne_1 = e_1, \quad (S_ne_2)(x) = x^2 + \frac{n}{x}.$$

□

Lemma 4.2. *The following equality holds*

$$\left(x - \frac{k}{n}\right)m_{n,k}(x,\alpha) = (1+\alpha x)\left(\frac{k+1}{n}m_{n,k+1}(x,\alpha) - \frac{k}{n}m_{n,k}(x,\alpha)\right).$$

The proof is a direct verification. We omit it.

Theorem 4.3. *For $x \geq 0$ and $i = [nx]$ we have*

a) $M_n^{(\alpha)}((t-x)^2;x) = \sum_{k=0}^\infty m_{n,k}(x,\alpha)\left(x-\frac{k}{n}\right)^2 = \frac{x(nx+\alpha)}{\alpha n}$;

b) $M_n^{(\alpha)}(|t-x|;x) = \sum_{k=0}^\infty m_{n,k}(x,\alpha)\left|x-\frac{k}{n}\right| = 2x\left(1+\frac{i}{\alpha}\right)m_{n,i}(x,\alpha)$.

Proof. a) is obtained by Lemma 4.1 and b) is derived from Lemma 4.2. □

Many classical positive linear operators can be obtained as special cases of (4.3)-(4.4). Let us list some examples.

For $\alpha = -n$,

$$m_{n,k}(x,-n) = \binom{n}{k} x^k (1-x)^{n-k}$$

and we obtain the Bernstein operator.

For $\alpha \to \infty$

$$\lim_{\alpha\to\infty} m_{n,k}(x,\alpha) = e^{-nx}\frac{(nx)^k}{k!}$$

and we obtain the Szász-Mirakjan operator (4.1).

For $\alpha = n$

$$m_{k,k}(x,n) = \binom{n+k-1}{k} x^k/(1+x)^{n+k}$$

and we obtain the Baskakov operator.

For $\alpha = nx$, $x > 0$,

$$m_{n,k}(x,nx) = 2^{-nx}\frac{(nx)^k}{2^k k!}$$

and we obtain the Lupaş operator [3]

$$(L_n f)(x) = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{2^k k!} f\left(\frac{k}{n}\right). \tag{4.4}$$

Lupaş established the Korovkin condition guaranteeing the approximation property. He remarked that the operator L_n have a form very similar to the Szász-Mirakjan operator (4.1) and invited to find further properties. O. Agratini investigated the operators of Lupaş. He derived an asymptotic formula and some quantitative estimates for the rate of convergence. In [1], U. Abel and M. Ivan give estimates for the rate of convergence and he derive the complete asymptotic expansion for this sequence of operators.

In the following, using the Theorem 4.2 we shall prove that this operator can be obtained by the composite of Rathore's operators (3.6) with Szász's operator (4.1).

Corollary 4.4. *If R_n is the Rathore's operator (3.6) then $L_n f = R_n(S_n f)$.*

Proof. The proof is obtained from Theorem 4.2 for $\alpha = nx$. □

Corollary 4.5. *The operator L_n can be written in the following manner*

$$(L_n f)(x) = \sum_{k=0}^{\infty}\frac{(nx)_k}{k!}\left[0,\frac{1}{n},\ldots,\frac{k}{n};f\right].$$

Proof. We apply Corollary 4.4, using for the Szász's operator the following formula

$$(S_n f)(x) = \sum_{k=0}^{\infty}\left[0,\frac{1}{n},\ldots,\frac{k}{n};f\right]x^k.$$

□

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