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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} *anniversary*

Existence and behavior of solutions of a system of quasilinear differential equations

ALMA OMERSPAHIĆ

ABSTRACT. This paper deals with behavior, approximation and stability of some solutions of the system of quasilinear differential equations. The behavior of solutions in the neighborhood of an arbitrary curve is considered, with extraordinary attention on some spacial cases. The obtained results contain an answer to the question on approximation as well as stability of solution whose existence is established. The errors of the approximation are defined by the function that can be sufficiently small. The theory of qualitative analysis of differential equations and topological retraction method are used.

1. INTRODUCTION

Let us consider the system of quasilinear differential equations

$$\dot{x} = A(x,t)\,x + F(x,t) \tag{1.1}$$

where $x(t) = (x_1(t), ..., x_n(t))^T,\ n \geq 2\,;\, t \in I = \langle a, \infty\rangle,\ a \in \mathbb{R}\,; D \subset \mathbb{R}^n$ is open set, $\Omega = D \times I$, $A(x,t) = (a_{ij}(x,t))_{n\times n}$ is the matrix-function with elements $a_{ij} \in C(\Omega, \mathbb{R})$ $(i,j = 1, ..., n)$. $F(x,t) = (f_1(x,t), ..., f_n(x,t))^T$ is the vector-function with elements $f_i \in C(\Omega, \mathbb{R})$, functions $f_i(x,t),\ i = 1, ..., n,$ satisfy the Lipschitz's condition with respect to the variable x.

Above conditions for the functions $a_{ij},\ f_i\ (i,j = 1, ..., n)$, grant the existence and unique solution of every Cauchy's problem for system (1.1) in Ω.

Let

$$\Gamma = \{(x,t) \in \Omega : x = \varphi(t),\ t \in I\}, \tag{1.2}$$

where $\varphi(t) = (\varphi_1(t), ..., \varphi_n(t)),\ \varphi_j(t) \in C^1(I, \mathbb{R}),$ is a certain curve in Ω.

In this paper the behavior of the solutions of system (1.1) in the neighbourhoods of curve Γ is considered. The qualitative analysis theory of differential equations and the topological retraction method of T. Važevski [7], are used.

2. NOTATIONS AND PRELIMINARIES

We shall consider the behavior of integral curves $(x(t), t), t \in I$, of system (1.1) with respect to the set

$$\omega = \{(x,t) \in \Omega : \sum_{i=1}^{n} (x_i - \varphi_i(t))^2 < r^2(t)\} \tag{2.3}$$

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where $r \in C^1(I, \mathbb{R}^+)$.

The boundary surface of set ω with respect to the set Ω is

$$W = \{(x,t) \in Cl\omega : B(x,t) := \sum_{i=1}^{n} (x_i - \varphi_i(t))^2 - r^2(t) = 0\}. \tag{2.4}$$

Let us denote the tangent vector field to an integral curve $(x(t), t),\ t \in I$, of (1.1) by T. The vector ∇B is the external normal on surface W. We have

$$\begin{aligned} T &= (\sum_{j=1}^{n} a_{1j}x_j + f_1, ..., \sum_{j=1}^{n} a_{ij}x_j + f_i, ..., \sum_{j=1}^{n} a_{nj}x_j + f_n, 1), \\ \frac{1}{2}\nabla B &= (x_1 - \varphi_1, ..., x_i - \varphi_i, ..., x_n - \varphi_n, -\sum_{i=1}^{n} (x_i - \varphi_i)\varphi_i' - rr'). \end{aligned}$$

By means of scalar product $P(x,t) = (\nabla B, T\ )$ on W, we shall establish the behavior of integral curve of (1.1) with respect to the set ω.

Let us denote by $S^n(I)$ a class of solutions $(x(t), t)$ of the system (1.1) defined on I, which depends on n parameters. We shall simply say that the class of solutions $S^n(I)$ belongs to the set ω if graphs of functions in $S^n(I)$ are contained in ω. In that case we shall write $S^n(I) \subset \omega$. For $p = 0$ we have the notation $S^0(I)$, which means that there exists at least one solution $(x(t), t)$ on I of the system (1.1),whose graph belongs to the set ω.

The results of this paper are based on the following Lemmas (see [5], [6])

Lemma 2.1. *If, for the system* (1.1)*, the scalar product* $P(x,t) < 0$ *on* W*, then the system* (1.1) *has a class of solutions* $S^n(I)$ *belonging to the set* ω *for all* $t \in I$*, i.e.,* $S^n(I) \subset \omega$.

Lemma 2.2. *If, for the system* (1.1)*, the scalar product* $P(x,t) > 0$ *on* W*, then the system* (1.1) *has at least one solution on* I *whose graph belongs to the set* ω *for all* $t \in I$*, i.e.,* $S^0(I) \subset \omega$.

3. Main results

Theorem 3.1. *Let the curve* Γ *is* t*-axis and* $r \in C^1(I, \mathbb{R}^+)$.

a) If the conditions

$$\sum_{i=1}^{n} f_i(x,t)\, x_i < r(t)\, r'(t)\,, \tag{3.5}$$

$$\sum_{j=1 (j\neq i)}^{n} |a_{ij}(x,t) + a_{ji}(x,t)| \leq -2\, a_{ii}(x,t)\,, \tag{3.6}$$

are satisfied on W*, then the system* (1.1) *has a class of solutions* $S^n(I) \subset \omega$.

b) If the conditions

$$\sum_{i=1}^{n} f_i(x,t)\, x_i > r(t)\, r'(t)\,, \tag{3.7}$$

$$\sum_{j=1 (j\neq i)}^{n} |a_{ij}(x,t) + a_{ji}(x,t)| \leq 2\, a_{ii}(x,t) \tag{3.8}$$

are satisfied on W*, then the system* (1.1) *has at least one solution whose graph belongs to the set* ω*, i.e.* $S^0(I) \subset \omega$.

Proof. According to the conditions of theorem the following estimates for

$$P(x,t) = \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_{ij} + a_{ji}) x_i x_j + \sum_{i=1}^{n} f_i x_i - r r' \tag{3.9}$$

on W are valid in the cases a) and b) respectively:

a)

$$\begin{aligned} P(x,t) \quad &\leq \quad \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{i=1}^{n-1} \sum_{j+1}^{n} \frac{1}{2} |a_{ij} + a_{ji}| (x_i^2 + x_j^2) + \sum_{i=1}^{n} f_i x_i - r r' \\ &= \quad \sum_{i=1}^{n} (a_{ii} + \sum_{j=1 (j\neq i)}^{n} \frac{1}{2} |a_{ij} + a_{ji}|) x_i^2 + \sum_{i=1}^{n} f_i x_i - r r' < 0. \end{aligned}$$

b)

$$\begin{aligned} P(x,t) \quad &\geq \quad \sum_{i=1}^{n} a_{ii} x_i^2 - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{2} |a_{ij} + a_{ji}| (x_i^2 + x_j^2) + \sum_{i=1}^{n} f_i x_i - r r' \\ &= \quad \sum_{i=1}^{n} (a_{ii} - \sum_{j=1 (j\neq i)}^{n} \frac{1}{2} |a_{ij} + a_{ji}|) x_i^2 + \sum_{i=1}^{n} f_i x_i - r r' > 0\,. \end{aligned}$$

According to the given Lemmas the above estimates for $P(x,t)$ imply the statements of the theorem. □

Theorem 3.2. *Let* Γ *is an arbitrary curve in* $\Omega,\ r \in C^1(I, \mathbb{R}^+)$ *and* $u \in C(\Omega)$.

a) If the conditions

$$\sum_{i=1}^{n} |\sum_{j=1}^{n} a_{ij}(x,t) \varphi_j + f_i(x,t) - \varphi_i'| \leq u(x,t)\, r(t) + r'(t), \tag{3.10}$$

$$\sum_{j=1 (j\neq i)}^{n} \frac{1}{2} |a_{ij}(x,t) + a_{ji}(x,t)| < -a_{ii}(x,t) - u(x,t)\,,\ i = 1, ..., n\,, \tag{3.11}$$

are satisfied on W*, then the system* (1.1) *has a class of solutions* $S^n(I) \subset \omega$.

b)If the conditions

$$\sum_{i=1}^{n} |\sum_{j=1}^{n} a_{ij}(x,t)\, \varphi_j + f_i(x,t) - \varphi_i'| \leq u(x,t)\, r(t) - r'(t), \tag{3.12}$$

$$\sum_{j=1 (j\neq i)}^{n} \frac{1}{2} |a_{ij}(x,t) + a_{ji}(x,t)| < a_{ii}(x,t) - u(x,t)\,,\ i = 1, ..., n \tag{3.13}$$

are satisfied on W*, then the system* (1.1) *has at least one solution whose graph belongs to the set* ω*, i.e.* $S^0(I) \subset \omega$.

Proof. For the scalar product $P(x,t) = (\frac{1}{2}\nabla B, T)$ we have:

$$P(x,t) = \sum_{i=1}^{n} a_{ii}(x_i - \varphi_i)^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_{ij} + a_{ji})(x_i - \varphi_i(x_j - \varphi_j) + \\ + \sum_{i=1}^{n}[(x_i - \varphi_i)(\sum_{j=1}^{n} a_{ij}\varphi_j + f_i - \varphi_i')] - rr'$$

In view of $(3.10) - (3.13)$, the following estimates for $P(x,t)$ on W are valid:

a)

$$\begin{aligned} P(x,t) &\leq \sum_{i=1}^{n}[a_{ii} + \sum_{j=1(j\neq i)}^{n} \frac{1}{2}|a_{ij} + a_{ji}|](x_i - \varphi_i)^2 + \\ &\qquad + r\sum_{i=1}^{n}|\sum_{j=1}^{n} a_{ij}\varphi_j + f_i - \varphi_i'| - rr' \\ &< -ur^2 + ur^2 + rr' - rr' = 0. \end{aligned}$$

b)

$$\begin{aligned} P(x,t) &\geq \sum_{i=1}^{n}[a_{ii} - \sum_{j=1(j\neq i)}^{n} \frac{1}{2}|a_{ij} + a_{ji}|](x_i - \varphi_i)^2 - \\ &\qquad - r\sum_{i=1}^{n}|\sum_{j=1}^{n} a_{ij}\varphi_j + f_i - \varphi_i'| - rr' \\ &> ur^2 + r(-ur + r') - rr' = 0. \end{aligned}$$

According to Lemma 2.1. and Lemma 2.2., the above estimates for $P(x,t)$ imply the statements of the theorem. □

4. Applications

For the model, which is known in plasma physics,

$$\begin{aligned} \dot{x}_1 &= -\nu_1 x_1 + M_1 x_2 x_3 \\ \dot{x}_2 &= -\nu_2 x_2 + M_2 x_1 x_3 \\ \dot{x}_3 &= -\nu_3 x_3 + M_3 x_1 x_2 \end{aligned} \tag{4.14}$$

where $\nu_i, M_i \in \mathbb{R}, \nu_i > 0$, we have:

Theorem 4.3. *Let* $r \in C^1(I, \mathbb{R}^+)$ *and* $\nu = \min\{\nu_1, \nu_2, \nu_3\}$. *If*

$$|M_1 + M_2 + M_3| \leq \frac{\nu}{r} + \frac{r'}{r^2}\,,\ t \in I\,, \tag{4.15}$$

then the system (4.14) *has a class of solutions* $S^3(I) \subset \omega$, *where*

$$\omega = \{(x_1, x_2, x_3, t) \in \mathbb{R}^3 \times I : x_1^2 + x_2^2 + x_3^2 < r^2(t)\}. \tag{4.16}$$

Proof. For the scalar product$P(x,t) = (\frac{1}{2}\nabla B, T)$ on W we have

$$\begin{aligned} P(x,t) &= (-\nu_1 x_1 + M_1 x_2 x_3)x_1 + (-\nu_2 x_2 + M_2 x_2 x_2)x_2 + \\ &\quad +(-\nu_3 x_3 + M_3 x_1 x_2)x_3 - rr' \\ &= -\nu_1 x_1^2 - \nu_2 x_2^2 - \nu_3 x_3^2 + x_1 x_2 x_3 (M_1 + M_2 + M_3) - rr' \\ &\leq -\nu(x_1^2 + x_2^2 + x_3^2) + |x_1 x_2 x_3|\,|M_1 + M_2 + M_3| - rr' \\ &< -\nu r^2 + r^3 |M_1 + M_2 + M_3| - rr' \leq 0\,. \end{aligned}$$

According to Lemma 2.1. we have $S^3(I) \subset \omega$. □

Example 4.1. For the system (4.14) and the initial condition

$$x_1^2(t_0) + x_2^2(t_0) + x_3^2(t_0) \leq \alpha^2 e^{-2\alpha^2 t_0}, \tag{4.17}$$

we can prove the following:

All solutions $(x_1(t), x_2(t), x_3(t))$ of system (4.14) which satisfy the initial condition (4.17) satisfy the condition

$$x_1^2(t) + x_2^2(t) + x_3^2(t) \leq \alpha^2 e^{-2\alpha^2 t},\ for\ t > t_0 \geq \frac{1}{\alpha^2}\ln\left(\frac{|M_1 + M_2 + M_3|}{\nu - \alpha}\right). \tag{4.18}$$

This result follows from Theorem 4.3., with $r(t) = \alpha e^{-\alpha^2 t},\ 0 < \alpha < \nu$.

The Lorenz model, which plays an important role in chemical kinetics, is described by the equations:

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= b x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= -c x_3 + x_1 x_2 \end{aligned} \tag{4.19}$$

where a, b, c are given real parameters.

Theorem 4.4. *Let* $r \in C^1(I, \mathbb{R}^+),\ a, c \in \mathbb{R}^+,\ p = \min\{1, a, c\}$. *If*

$$|a + b| < 2(p + \frac{r'}{r})\,,\ t \in I, \tag{4.20}$$

then the system (4.19) *has a class of solutions* $S^3(I) \subset \omega$, *where*

$$\omega = \{(x_1, x_2, x_3, t) \in \mathbb{R}^3 \times I : x_1^2 + x_2^2 + x_3^2 < r^2(t)\}. \tag{4.21}$$

Proof. For the scalar product $P(x,t) = (\frac{1}{2}\nabla B, T)$ on W we have

$$\begin{aligned} P(x,t) &= a(x_2 - x_1)x_1 + (b x_1 - x_2 - x_1 x_3)x_2 + (-c x_3 + x_1 x_2)x_3 - rr' \\ &= (a+b)x_1 x_2 - (a x_1^2 + x_2^2 + c x_3^2) - rr' \\ &\leq \frac{1}{2}|a+b|(x_1^2 + x_2^2) - p(x_1^2 + x_2^2 + x_3^2) - rr' \\ &\leq \frac{1}{2}|a+b|\,r^2 - p r^2 - rr' < 0\,. \end{aligned}$$

According to Lemma 2.1. we have $S^3(I) \subset \omega$. □

Example 4.2. For the Lorenz model (4.19) and the condition

$$x_1^2(t_0) + x_2^2(t_0) + x_3^2(t_0) \leq \alpha^2 e^{-6}, \;\; \alpha \in \mathbb{R}^+, \tag{4.22}$$

we can prove the following:

If parameters of system (4.19) satisfy the relations

$$a \geq 1, \; c \geq 1, \; |a+b| \leq \frac{5}{3}, \tag{4.23}$$

then all solutions $(x_1(t), x_2(t), x_3(t))$ of system (4.19), which satisfy the initial condition (4.22) satisfy the condition

$$x_1^2(t) + x_2^2(t) + x_3^2(t) \leq \alpha^2 e^{-2\sqrt{t}}, \;\; for\; t > 9\,. \tag{4.24}$$

This result follows from Theorem 4.4., with $r(t) = \alpha e^{-\sqrt{t}}$, $\alpha \in \mathbb{R}^+$.

Remark 4.1. The obtained results also give the possibility to discuss the stability (instability) of solutions of the system (1.1). For example, under the conditions of Theorem 3.1, case a), every solution of (1.1) with initial value in ω is r-stable (stable with the function of stability r), if $r(t)$ tends to zero as $t \to \infty$ and $r'(t) < 0, \; t \in I$. However, if we consider the Theorem 3.1, case b), then established solution in ω is r-instable in case where $r'(t) > 0, \; t \in I$. The same conclusion holds for other theorems.

Remark 4.2. The obtained results also contain an answer to the question on approximation of solutions $x(t)$ whose existence is established. The errors of the approximation are defined by the functions $r(t)$ which can be arbitrary small $\forall t \in I$.

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UNIVERSITY OF SARAJEVO
MECHANICAL ENGINEERING FACULTY
VILSONOVO SETALISTE 9
71000 SARAJEVO
BOSNIA AND HERZEGOVINA
E-mail address: alma.omerspahic@mef.unsa.ba